MA 523
EXAM 1

You may not use notes, books, calculators, laptops, phones, or other devices of any kind.

(1) Begin each problem on a clean sheet of paper. (2) Write only on one side of each sheet. (3) Show all of your work—credit is generally not given for answers only. (4) Arrange your sheets in numerical order, fold vertically, and put your name on the outside sheet before handing in your exam.

1. Define the following terms.

(a) Linear function $f(\mathbf{v})$ on a vector space $\mathcal{V}$.

(b) Basis for a vector space.

(c) Dimension of a vector space.

(d) Sherman-Morrison formula for $(\mathbf{A}_{n\times n} + \mathbf{c}\mathbf{d}^T)^{-1}$, where $\mathbf{c}, \mathbf{d} \in \mathbb{R}^{n\times 1}$.

(e) The rank and range of $\mathbf{A} \in \mathbb{R}^{m\times n}$.

2. (a) Must a homogeneous system of m equations in n unknowns where $m < n$ must always possess an infinite number of solutions? Why?

(b) Must a nonhomogeneous system of m equations in n unknowns where $m < n$ always possess an infinite number of solutions. Why?

3. Let $\mathcal{Z}_n$ denote the set of all $n \times n$ real matrices having $trace\,(\mathbf{A}) = 0$.

(a) Prove that $\mathcal{Z}_n$ is a subspace of $\mathbb{R}^{n\times n}$.

(b) For 2×2 matrices, determine $\dim \mathcal{Z}_2$, and prove that your answer is correct.

4. Solve the system $\begin{aligned} -10x + 10^5 y &= 10^5 \\ x + \quad y &= 2 \end{aligned}$ using 3-digit floating-point arithmetic with the following rules.

(a) First use partial pivoting *without scaling* to find the 3-digit solution.

(b) Now use partial pivoting *with row scaling* to find the 3-digit solution.

5. Write $\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}^{-1}$ as a product of elementary matrices. You must show your work and explain your reasoning to receive any credit.

SOLUTIONS

1. (a) A linear function on $\mathcal{V}$ is a function f from $\mathcal{V}$ into $\mathcal{V}$ such that $\mathbf{f}(\alpha\mathbf{v}_1+\mathbf{v}_2)=\alpha\mathbf{f}(\mathbf{v}_1)+\mathbf{f}(\mathbf{v}_2)$ for all $\mathbf{v}_1,\ \mathbf{v}_2\in\mathcal{V}$ and for all scalars α in the field over which $\mathcal{V}$ is defined.

(b) A basis for a vector space $\mathcal{V}$ is defined to be a linearly independent spanning set for $\mathcal{V}$.

(c) $\dim\mathcal{V}$ is defined to be the number of vectors in any basis for $\mathcal{V}$.

(c) If $\mathbf{A}_{n\times n}$ is nonsingular and if $\mathbf{c}$ and $\mathbf{d}$ are $n\times 1$ columns such that $1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}\neq 0$, then the sum $\mathbf{A}+\mathbf{c}\mathbf{d}^T$ is nonsingular, and

$$\left(\mathbf{A}+\mathbf{c}\mathbf{d}^T\right)^{-1}=\mathbf{A}^{-1}-\frac{\mathbf{A}^{-1}\mathbf{c}\mathbf{d}^T\mathbf{A}^{-1}}{1+\mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}.$$

(e) The rank of a matrix $\mathbf{A}$ is defined to be

$$\begin{aligned} rank\,(\mathbf{A}) &= \text{number of pivots}\\ &= \text{number of nonzero rows in an echelon form derived from } \mathbf{A}\\ &= \text{number of basic columns in } \mathbf{A}, \end{aligned}$$

where the basic columns of $\mathbf{A}$ are those columns in $\mathbf{A}$ that correspond to the pivot positions. The range of $\mathbf{A}$ is $R(\mathbf{A})=\{\mathbf{A}\mathbf{x}\,|\,\mathbf{x}\in\mathbb{R}^n\}$, the set of all possible images.

2. (a) Yes, because the number of free variables is $n-r$, and $r=rank\,(\mathbf{A})\le m<n \implies n-r>0$.

(b) No, the system may be inconsistent, so there may be no solutions at all.

3. (a) $\mathcal{Z}_n$ is closed with respect to addition (i.e., $\mathbf{A},\mathbf{B}\in\mathcal{Z}_n \implies \mathbf{A}+\mathbf{B}\in\mathcal{Z}_n$) because if $trace\,(\mathbf{A})=0$ and $trace\,(\mathbf{B})=0$, then, by linearity of the trace function, $trace\,(\mathbf{A}+\mathbf{B})=trace\,(\mathbf{A})+trace\,(\mathbf{B})=0+0=0$. Similarly, $\mathcal{Z}_n$ is closed with respect to scalar multiplication because if $trace\,(\mathbf{A})=0$, then the linearity of trace gives $trace\,(\alpha\mathbf{A})=\alpha\times trace\,(\mathbf{A})=0$.

(b) A basis for the space of 2×2 matrices with zero trace is $\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix},\begin{pmatrix}0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$ because every 2×2 matrix with zero trace is a combination of these three (i.e., they are a spanning set), and this set of three matrices is linearly independent (because $\alpha_1\mathbf{B}_1+\alpha_2\mathbf{B}_2+\alpha_3\mathbf{B}_3=\mathbf{0} \implies \alpha_1=\alpha_2=\alpha_3=0$). Therefore, $\dim\mathcal{Z}_2=3$.

4. (a) No interchange is needed because $1<|-10|$. Proceed with 3-digit elimination.

$$\left(\begin{array}{cc|c}-10&10^5&10^5\\1&1&2\end{array}\right)\ R_2+10^{-1}R_1\longrightarrow\left(\begin{array}{cc|c}-10&10^5&10^5\\0&10^4&10^4\end{array}\right)$$

because

$$fl(1+10^4)=fl(.10001\times 10^5)=.100\times 10^5=10^4$$

and

$$fl(2+10^4)=fl(.10002\times 10^5)=.100\times 10^5=10^4.$$

Back substitution yields $x=0$ and $y=1$.

(b) The row-scaled system is $\left(\begin{array}{cc|c} -10^{-4} & 1 & 1 \\ 1 & 1 & 2 \end{array}\right)$, and partial pivoting yields

$$\left(\begin{array}{cc|c} 1 & 1 & 2 \\ -10^{-4} & 1 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right)$$

because

$$fl(10^{-4}+1)=1 \quad \text{and} \quad fl(2\times 10^{-4}+1)=1$$

Back substitution yields $x=1$ and $y=1$.

5. By reducing $\mathbf{A}$ to $\mathbf{I}$ with row operations

$$\mathbf{A}=\begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that $\mathbf{E}_2\mathbf{E}_1\mathbf{A}=\mathbf{I}$ where $\mathbf{E}_1=\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$ and $\mathbf{E}_2=\begin{pmatrix} 1 & 0 \\ 0 & 1/3 \end{pmatrix}$. By definition of the inverse, it must be the case that $\mathbf{A}^{-1}=\mathbf{E}_2\mathbf{E}_1$.