MA 523
EXAM 1

Directions: (1) Begin each problem on a clean sheet of paper. (2) Write only on one side of each sheet. (3) Show all of your work—credit is generally not given for answers only. (4) Arrange your sheets in numerical order, fold vertically, and put your name on the outside sheet before handing in your exam.

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1. Define the following terms or formulas—be concise and precise.

(a) Linear function on a vector space $\mathcal{V}$.
(b) Spanning set for a finite dimensional vector space $\mathcal{V}$.
(c) LU factorization of a nonsingular matrix $\mathbf{A}$.
(d) Sherman-Morrison formula for inverting a rank-one updated matrix $\mathbf{A} + \mathbf{c}\mathbf{d}^T$.

2. With no scaling, compute the 3-digit solution of $\left\{ \begin{aligned} 10^{-3}x - y &= 1, \\ x + y &= 0. \end{aligned} \right\}$ without partial pivoting and with partial pivoting.

3. Among all solutions that satisfy the homogeneous system $\left\{ \begin{aligned} x + 2y + z &= 0, \\ 2x + 4y + z &= 0, \\ x + 2y - z &= 0, \end{aligned} \right\}$ determine those that also satisfy the nonlinear constraint $y - xy = 2z$.

4. Determine $\dim N(\mathbf{A}) \cap R(\mathbf{B})$ for $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}$.

5. Suppose that $\mathbf{A}_{m\times n}\mathbf{x}_{n\times 1} = \mathbf{b}_{m\times 1}$ is a consistent system of linear equations, where $\mathbf{b} \neq \mathbf{0}$. Either prove or disprove that the set $\mathcal{S}$ of all solutions to $\mathbf{Ax} = \mathbf{b}$ is a subspace of $\Re^n$.

6. Suppose that $\mathbf{A}$ is a square matrix such that $\mathbf{A} \geq \mathbf{0}$ (i.e., all entries are nonnegative numbers). Prove that if $\lim_{n\to\infty} \mathbf{A}^n = \mathbf{0}$, then $(\mathbf{I} - \mathbf{A})^{-1}$ does not have negative entries.

SOLUTIONS

1. (a) A linear function on $\mathcal{V}$ is a function $\mathbf{T}$ from $\mathcal{V}$ into $\mathcal{V}$ such that $\mathbf{T}(\alpha\mathbf{v}_1 + \mathbf{v}_2) = \alpha\mathbf{T}(\mathbf{v}_1) + \mathbf{T}(\mathbf{v}_2)$ for all $\mathbf{v}_1, \mathbf{v}_2 \in \mathcal{V}$ and for all scalars α in the field over which $\mathcal{V}$ is defined.

(b) $\mathcal{S}$ is a spanning set for $\mathcal{V}$ whenever each vector in $\mathcal{V}$ is a linear combination of vectors from $\mathcal{S}$.

(c) If $\mathbf{A}$ is an $n \times n$ matrix such that a zero pivot is never encountered when applying gaussian elimination with Type III operations, then $\mathbf{A}$ can be factored as the product $\mathbf{A} = \mathbf{LU}$, where the following hold. (1) $\mathbf{L}$ is lower triangular and $\mathbf{U}$ is upper triangular. (2) $\ell_{ii} = 1$ and $u_{ii} \neq 0$ for each $i = 1, 2, \ldots, n$. (3) Below the diagonal of $\mathbf{L}$, the entry ℓ_{ij} is the multiple of row j that is subtracted from row i in order to annihilate the (i, j)-position during gaussian elimination. (4) $\mathbf{U}$ is the final result of gaussian elimination applied to $\mathbf{A}$.

(d) If $\mathbf{A}_{n\times n}$ is nonsingular and if $\mathbf{c}$ and $\mathbf{d}$ are $n \times 1$ columns such that $1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c} \neq 0$, then the sum $\mathbf{A} + \mathbf{cd}^T$ is nonsingular, and

$$\left(\mathbf{A} + \mathbf{cd}^T\right)^{-1} = \mathbf{A}^{-1} - \frac{\mathbf{A}^{-1}\mathbf{cd}^T\mathbf{A}^{-1}}{1 + \mathbf{d}^T\mathbf{A}^{-1}\mathbf{c}}.$$

2. Without PP: $(0, -1)$ With PP: $(1, -1)$ Exact: $\left(\frac{1}{1.001}, \frac{-1}{1.001}\right)$

3. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1/2 \\ 0 \end{pmatrix}$

4. $\dim N(\mathbf{A}) \cap R(\mathbf{B}) = rank(\mathbf{B}) - rank(\mathbf{AB}) = 2 - 1 = 1.$

5. Not a subspace because $\mathcal{S}$ is not closed under addition (i.e., the sum of two solutions is not necessarily a solution). *Proof.* $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ implies $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{Ay} = \mathbf{b}$, but $\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{Ax} + \mathbf{Ay} = \mathbf{b} + \mathbf{b} = 2\mathbf{b} \neq \mathbf{b}$ because $\mathbf{b} \neq \mathbf{0}$.

6. $\lim_{n\to\infty} \mathbf{A}^n = \mathbf{0}$ implies that the Neumann series $\sum_{k=0}^{\infty} \mathbf{A}^k$ converges to $(\mathbf{I} - \mathbf{A})^{-1}$. Furthermore, $\mathbf{A} \geq \mathbf{0}$ implies $\mathbf{A}^k \geq \mathbf{0}$ for all k, so $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \cdots \geq \mathbf{0}$.