# MA 523
## EXAM 2

**Directions:** (1) Begin each problem on a clean sheet of paper. (2) Write only on one side of each sheet. (3) Show all of your work—credit is generally not given for answers only. (4) Arrange your sheets in numerical order, fold vertically, and put your name on the outside sheet before handing in your exam.

**1.** Determine $\dim N(\mathbf{A}) \cap R(\mathbf{B})$ for $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}$.
(Explain your reasoning and show your work.)

**2.** A small company has been in business for three years and has recorded annual profits (in thousands of dollars) as follows.

| Year | 1 | 2 | 3 |
|---|---|---|---|
| Profit | 7 | 4 | 3 |

Assuming that there is a linear trend in the declining profits, predict the year and the month in which the company begins to lose money. (Explain your reasoning and show your work.)

**3.** Determine the angle between $\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
(Explain your reasoning and show your work.)

**4.** Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ be the nonzero singular values of $\mathbf{A}_{m\times n}$. Explain why the Frobenius matrix norm is $\|\mathbf{A}\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_r^2}$.

**5.** Find a basis for the orthogonal complement of $\mathcal{M} = span \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \\ 6 \end{pmatrix} \right\}$.

**6.** If $\{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ is an orthonormal basis for an inner-product space $\mathcal{V}$, explain why

$$\langle \mathbf{x} | \mathbf{y} \rangle = \sum_i \langle \mathbf{x} | \mathbf{u}_i \rangle \langle \mathbf{u}_i | \mathbf{y} \rangle \quad \text{for every } \mathbf{x}, \mathbf{y} \in \mathcal{V}.$$

# SOLUTIONS

**1.** $\dim N(\mathbf{A}) \cap R(\mathbf{B}) = rank(\mathbf{B}) - rank(\mathbf{AB}) = 2 - 1 = 1.$

**2.** Look for the line $p = \alpha + \beta t$ that comes closest to the data in the least squares sense. That is, find the least squares solution for the system $\mathbf{Ax} = \mathbf{b}$, where

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 7 \\ 4 \\ 3 \end{pmatrix}.$$

Set up normal equations $\mathbf{A}^T\mathbf{Ax} = \mathbf{A}^T\mathbf{b}$ to get

$$\begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 14 \\ 24 \end{pmatrix} \implies \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 26/3 \\ -2 \end{pmatrix} \implies p = (26/3) - 2t.$$

Setting $p = 0$ gives $t = 13/3$. In other words, we expect the company to begin losing money on May 1 of year five.

**3.** $\cos\theta = \langle \mathbf{I}|\mathbf{B}\rangle / \|\mathbf{I}\|\,\|\mathbf{B}\| = trace\left(\mathbf{I}^T\mathbf{B}\right)/(\sqrt{trace\,(\mathbf{I}^T\mathbf{I})}\sqrt{trace\,(\mathbf{B}^T\mathbf{B})}\,) = 2/(\sqrt{2}\sqrt{4}) = 1/\sqrt{2} \implies \theta = \pi/4.$

**4.** If $\mathbf{A} = \mathbf{U}\begin{pmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T$ is an SVD where $\mathbf{D} = \text{diag}\,(\sigma_1, \sigma_2, \ldots, \sigma_r)$, then

$$\|\mathbf{A}\|_F^2 = trace\left(\mathbf{A}^T\mathbf{A}\right) = trace\mathbf{V}\begin{pmatrix} \mathbf{D}^2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\mathbf{V}^T = trace\left(\mathbf{D^2}\right) = \sigma_1^2 + \cdots + \sigma_r^2.$$

**5.** If $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ 0 & 1 \\ 3 & 6 \end{pmatrix}$, then $R(\mathbf{A}) = \mathcal{M}$, so $\mathcal{M}^\perp = N\left(\mathbf{A}^T\right)$. Using row operations, a basis for $N\left(\mathbf{A}^T\right)$ is computed to be $\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$.

**6.** Use the Fourier expansion $\mathbf{y} = \sum_i \langle \mathbf{u}_i|\mathbf{y}\rangle\,\mathbf{u}_i$ together with the linearity properties of an inner product to write

$$\langle \mathbf{x}|\mathbf{y}\rangle = \left\langle \mathbf{x} \,\middle|\, \sum_i \langle \mathbf{u}_i|\mathbf{y}\rangle\,\mathbf{u}_i \right\rangle = \sum_i \langle \mathbf{x}|\,\langle \mathbf{u}_i|\mathbf{y}\rangle\,\mathbf{u}_i\rangle = \sum_i \langle \mathbf{u}_i|\mathbf{y}\rangle\,\langle \mathbf{x}|\mathbf{u}_i\rangle\,.$$