MA 523
EXAM 3

Directions: (1) Begin each problem on a clean sheet of paper. (2) Write only on one side of each sheet. (3) Show all of your work—credit is generally not given for answers only. (4) Arrange your sheets in numerical order, fold vertically, and put your name on the outside sheet before handing in your exam. (4) You may not use notes, books, calculators, laptops, phones, or other devices of any kind.

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1. Define the following terms.
(i) Algebraic and geometric multiplicity of an eigenvalue.
(ii) Gerschgorin circles and Gerschgorin's Theorem.
(iii) Normal matrix.
(iv) Projection of $\mathbf{v} \in \mathcal{V}$ onto $\mathcal{X} \subseteq \mathcal{V}$ along $\mathcal{Y} \subseteq \mathcal{V}$.

2. Prove that $\mathbf{I}_{n\times n} + \mathbf{P}^*\mathbf{P}$ is nonsingular for every $k \times n$ matrix $\mathbf{P}$, where $1 \leq k \leq n$.

3. Explain why the set $\mathcal{Z}_2$ of all 2×2 real matrices having $trace\,(\mathbf{A}) = 0$ is a subspace of $\mathbb{R}^{2\times 2}$, and then determine $\dim \mathcal{Z}_2$. (Prove what you claim.)

4. Determine $\dim N(\mathbf{A}) \cap R(\mathbf{B})$ for $\mathbf{A} = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 3 & 1 & -4 \\ -1 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix}$.

5. Determine all values of ξ for which $\mathbf{A} = \begin{pmatrix} \xi & 2 & 0 \\ 1 & \xi & 1 \\ 0 & 1 & \xi \end{pmatrix}$ fails to have an LU factorization.

6. (a) For the $n \times n$ identity matrix $\mathbf{I}$, prove that $\|\mathbf{I}\| = 1$ for all induced matrix norms.
(b) Give an example to show that $\|\mathbf{I}\| = 1$ is not true for all matrix norms.

7. Using the standard inner product for matrices, compute the Fourier expansion of $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ with respect to the following orthonormal basis for $\mathbb{R}^{2\times 2}$.

$$\mathcal{B} = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}$$

8. Let $\mathcal{M} = span\,\{\mathbf{u}\}$ for $\mathbf{u} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. For $\mathbf{b} = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, determine the orthogonal projection of $\mathbf{b}$ onto $\mathcal{M} = span\,\{\mathbf{u}\}$, and then determine the orthogonal projection of $\mathbf{b}$ onto $\mathcal{M}^\perp$.

9. Find the singular values of $\mathbf{A} = \begin{pmatrix} 3 \\ 4 \\ \sqrt{11} \end{pmatrix}$.

10. Find all eigenvalues and eigenvectors of $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Is $\mathbf{A}$ similar to a diagonal matrix? Explain why.

SOLUTIONS

1. (i) The algebraic multiplicity of λ is the number of times λ is repeated as a root of the characteristic polynomial—i.e., $alg\ mult_{\mathbf{A}}(\lambda_i) = a_i$ if and only if $(x-\lambda_1)^{a_1}\cdots(x-\lambda_s)^{a_s} = 0$ is the characteristic equation for $\mathbf{A}$. The geometric multiplicity of λ is $\dim N(\mathbf{A}-\lambda\mathbf{I})$—i.e., $geo\ mult_{\mathbf{A}}(\lambda)$ is the maximal number of linearly independent eigenvectors associated with λ.

(ii) The Gerschgorin circles that are defined by the rows and columns of $\mathbf{A}\in\mathcal{C}^{n\times n}$ are

$$|z-a_{ii}|\le r_i, \quad \text{where} \quad r_i = \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| \text{ for } i = 1,2,\ldots,n.$$

and

$$|z-a_{jj}|\le c_j, \quad \text{where} \quad c_j = \sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| \text{ for } j = 1,2,\ldots,n.$$

Gerschgorin's theorem says that all eigenvalues of $\mathbf{A}$ are contained in the union $\mathcal{G}_r$ the row circles and also in the union $\mathcal{G}_c$ of the column circles so that $\sigma(\mathbf{A})\subset\mathcal{G}_r\cap\mathcal{G}_c$.

(iii) A square matrix $\mathbf{A}$ is defined to be normal when $\mathbf{A}^*\mathbf{A}=\mathbf{A}\mathbf{A}^*$, or equivalently, when $\mathbf{A}$ has a full set of orthonormal eigenvectors.

(iv) If $\mathcal{V}=\mathcal{X}\oplus\mathcal{Y}$, then there are unique vectors $\mathbf{x}\in\mathcal{X}$ and $\mathbf{y}\in\mathcal{Y}$ such that $\mathbf{v}=\mathbf{x}+\mathbf{y}$. The vector $\mathbf{x}$ is called the projection of $\mathbf{v}$ onto $\mathcal{X}$ along $\mathcal{Y}$.

2. Show that $\mathbf{I}+\mathbf{P}^*\mathbf{P}$ is nonsingular by showing that $N(\mathbf{I}+\mathbf{P}^*\mathbf{P})=\mathbf{0}$. This is done by observing that

$$\mathbf{x}\in N(\mathbf{I}+\mathbf{P}^*\mathbf{P}) \implies \mathbf{x}+\mathbf{P}^*\mathbf{P}\mathbf{x}=\mathbf{0} \implies \mathbf{x}^*\mathbf{x}+\mathbf{x}^*\mathbf{P}^*\mathbf{P}\mathbf{x}=0 \implies \mathbf{x}=\mathbf{0}$$

because $\mathbf{x}^*\mathbf{x}+\mathbf{x}^*\mathbf{P}^*\mathbf{P}\mathbf{x}=\|\mathbf{x}\|^2+\|\mathbf{P}\mathbf{x}\|^2=0$ if and only if $\mathbf{x}=\mathbf{0}$ and $\mathbf{P}\mathbf{x}=\mathbf{0}$.

3. (a) $\mathcal{Z}_n$ is closed with respect to addition (i.e., $\mathbf{A},\mathbf{B}\in\mathcal{Z}_n \implies \mathbf{A}+\mathbf{B}\in\mathcal{Z}_n$) because if $trace(\mathbf{A})=0$ and $trace(\mathbf{B})=0$, then, by linearity of the trace function, $trace(\mathbf{A}+\mathbf{B})=trace(\mathbf{A})+trace(\mathbf{B})=0+0=0$. Similarly, $\mathcal{Z}_n$ is closed with respect to scalar multiplication because if $trace(\mathbf{A})=0$, then the linearity of trace gives $trace(\alpha\mathbf{A})=\alpha\times trace(\mathbf{A})=0$.

(b) A basis for the space of 2×2 matrices with zero trace is $\left\{\begin{pmatrix}1&0\\0&-1\end{pmatrix}, \begin{pmatrix}0&1\\0&0\end{pmatrix}, \begin{pmatrix}0&0\\1&0\end{pmatrix}\right\}$ because every 2×2 matrix with zero trace is a combination of these three (i.e., they are a spanning set), and this set of three matrices is linearly independent (because $\alpha_1\mathbf{B}_1+\alpha_2\mathbf{B}_2+\alpha_3\mathbf{B}_3=\mathbf{0} \implies \alpha_1=\alpha_2=\alpha_3=0$). Therefore, $\dim\mathcal{Z}_2=3$.

4. $\dim N(\mathbf{A})\cap R(\mathbf{B}) = rank(\mathbf{B}) - rank(\mathbf{AB}) = 2-1=1.$

5. Looking for zero pivots when reducing by using only type III row operations shows that $\xi=0,\ \pm\sqrt{2},\ \pm\sqrt{3}$.

6. (a) $\|\mathbf{I}\|_\star = \max_{\|\mathbf{x}\|_\star=1}\|\mathbf{I}\mathbf{x}\|_\star = \max_{\|\mathbf{x}\|_\star=1}\|\mathbf{x}\|_\star = 1.$

(b) It's false for the Frobenius norm because $\|\mathbf{I}\| = \sqrt{n}$.

7. The Fourier coefficients $\langle \mathbf{U}_i | \mathbf{A} \rangle = trace(\mathbf{U}_i^T \mathbf{A})$ are

$$\langle \mathbf{U}_1 | \mathbf{A} \rangle = \frac{2}{\sqrt{2}}, \quad \langle \mathbf{U}_2 | \mathbf{A} \rangle = 0, \quad \langle \mathbf{U}_3 | \mathbf{A} \rangle = 1, \quad \langle \mathbf{U}_4 | \mathbf{A} \rangle = 1,$$

so the Fourier expansion of $\mathbf{A}$ is $\mathbf{A} = (2/\sqrt{2})\mathbf{U}_1 + \mathbf{U}_3 + \mathbf{U}_4$.

8. $\mathbf{P}_{\mathcal{M}} = \mathbf{u}\mathbf{u}^T/(\mathbf{u}^T\mathbf{u}) = (1/10)\begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$, and $\mathbf{P}_{\mathcal{M}^\perp} = \mathbf{I} - \mathbf{P}_{\mathcal{M}} = (1/10)\begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$, so $\mathbf{P}_{\mathcal{M}}\mathbf{b} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, and $\mathbf{P}_{\mathcal{M}^\perp}\mathbf{b} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$.

9. In general, the singular values are the positive square roots of the eigenvalues of $\mathbf{A}^T\mathbf{A}$, so in this case there is only one singular value, and it is $\sigma = \sqrt{36} = 6$.

10. $0 = det(\mathbf{A} - \lambda\mathbf{I}) = det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2$ means that $\lambda = 0$ is an eigenvalue of algebraic multiplicity two. The eigenvectors are solutions to $\mathbf{A}\mathbf{x} = \mathbf{0}$, which are $\mathbf{x} = x_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where x_1 is a free variable. $\mathbf{A}$ Is *not* similar to a diagonal matrix because there is not a set of two linearly independent eigenvectors, or equivalently, because $2 = \textit{alg mult}\,(0) \neq \textit{geo mult}\,(0) = 1$.