1. Define the following terms or state the following theorems.
   (a) Gerschgorin circles and Gerschgorin’s Theorem.  
   (b) Algebraic multiplicity of an eigenvalue. 
   (c) Geometric multiplicity of an eigenvalue. 
   (d) URV factorization. 
   (e) Projection of $v \in V$ onto $X \subseteq V$ along $Y \subseteq V$. 
   (f) Core-nilpotent decomposition.

2. Explain why every diagonally dominant matrix must be nonsingular.

3. Determine the singular values for $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, and then give an SVD for $A$.

4. Let $\mathcal{M} = \text{span} \{u\}$ for $u = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. For $b = \begin{pmatrix} 4 \\ 8 \end{pmatrix}$, determine the orthogonal projection of $b$ onto $\mathcal{M} = \text{span} \{u\}$, and then determine the orthogonal projection of $b$ onto $\mathcal{M}^\perp$.

5. Prove that if $A \in \mathbb{R}^{n \times n}$ is a diagonalizable matrix, then $\det(e^A) = e^{\text{trace}(A)}$.

6. Let $A_{m \times m}$ be a diagonalizable matrix with eigenvalues $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \cdots \geq |\lambda_k|$, and let $L = \lim_{n \to \infty} A^n/\lambda_1^n$. Prove that $\text{rank}(L) = a_1$, where $a_1$ is the algebraic multiplicity of $\lambda_1$. 
1. (a) The Gerschgorin circles that are defined by the rows and columns of \( \mathbf{A} \in \mathbb{C}^{n \times n} \) are

\[
|z - a_{ii}| \leq r_i, \quad \text{where} \quad r_i = \sum_{j \neq i}^{n} |a_{ij}| \quad \text{for} \quad i = 1, 2, \ldots, n.
\]

and

\[
|z - a_{jj}| \leq c_j, \quad \text{where} \quad c_j = \sum_{i \neq j}^{n} |a_{ij}| \quad \text{for} \quad j = 1, 2, \ldots, n.
\]

Gerschgorin’s theorem says that all eigenvalues of \( \mathbf{A} \) are contained in the union \( G_r \) of the row circles and also in the union \( G_c \) of the column circles so that \( \sigma(\mathbf{A}) \subseteq G_r \cap G_c \).

(b) The algebraic multiplicity of \( \lambda \) is the number of times \( \lambda \) is repeated as a root of the characteristic polynomial. In other words, \( \text{alg mult}_\mathbf{A}(\lambda_i) = a_i \) if and only if \((x - \lambda_1)^{a_1} \cdots (x - \lambda_s)^{a_s} = 0\) is the characteristic equation for \( \mathbf{A} \).

(c) The geometric multiplicity of \( \lambda \) is \( \dim \mathcal{N}(\mathbf{A} - \lambda \mathbf{I}) \). In other words, \( \text{geo mult}_\mathbf{A}(\lambda) \) is the maximal number of linearly independent eigenvectors associated with \( \lambda \).

(d) For each \( \mathbf{A} \in \mathbb{R}^{m \times n} \) of rank \( r \), there are orthogonal matrices \( \mathbf{U}_{m \times m} \) and \( \mathbf{V}_{n \times n} \) and a nonsingular matrix \( \mathbf{C}_{r \times r} \) such that

\[
\mathbf{A} = \mathbf{U} \mathbf{R} \mathbf{V}^T = \mathbf{U} \begin{pmatrix} \mathbf{C}_{r \times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{V}^T_{m \times n}.
\]

The first \( r \) columns in \( \mathbf{U} \) are an orthonormal basis for \( \mathcal{R}(\mathbf{A}) \). The last \( m - r \) columns of \( \mathbf{U} \) are an orthonormal basis for \( \mathcal{N}(\mathbf{A}^T) \). The first \( r \) columns in \( \mathbf{V} \) are an orthonormal basis for \( \mathcal{R}(\mathbf{A}^T) \). The last \( n - r \) columns of \( \mathbf{V} \) are an orthonormal basis for \( \mathcal{N}(\mathbf{A}) \).

(e) If \( \mathcal{V} = \mathcal{X} \oplus \mathcal{Y} \), then there are unique vectors \( \mathbf{x} \in \mathcal{X} \) and \( \mathbf{y} \in \mathcal{Y} \) such that \( \mathbf{v} = \mathbf{x} + \mathbf{y} \). The vector \( \mathbf{x} \) is called the projection of \( \mathbf{v} \) onto \( \mathcal{X} \) along \( \mathcal{Y} \).

(f) If \( \mathbf{A} \) is an \( n \times n \) singular matrix of index \( k \) such that \( \text{rank}(\mathbf{A}^k) = r \), then there exists a nonsingular matrix \( \mathbf{Q} \) such that

\[
\mathbf{Q}^{-1} \mathbf{A} \mathbf{Q} = \begin{pmatrix} \mathbf{C}_{r \times r} & \mathbf{0} \\ \mathbf{0} & \mathbf{N} \end{pmatrix}
\]

in which \( \mathbf{C} \) is nonsingular, and \( \mathbf{N} \) is nilpotent of index \( k \).

2. \( \mathbf{A}_{n \times n} \) is diagonally dominant whenever \( |a_{ii}| > \sum_{j \neq i}^{n} |a_{ij}| = r_i \) for each \( i = 1, 2, \ldots, n \). Gerschgorin’s theorem guarantees that all eigenvalues must be somewhere in the union of the disks centered at \( a_{ii} \) with radii \( r_i \). Consequently, the origin cannot be in any Gerschgorin disk when \( \mathbf{A} \) is diagonally dominant, so \( 0 \notin \sigma(\mathbf{A}) \), and hence \( \mathbf{A} \) cannot be singular.

3. There is only one singular value, which is \( \sqrt{\lambda(\mathbf{A}^T \mathbf{A})} = \sqrt{2} \). This means that the orthogonal matrix \( \mathbf{V} \) in \( \mathbf{A} = \mathbf{U} \mathbf{D} \mathbf{V}^T \) is \( 1 \times 1 \), so \( \mathbf{V} = [1] \). The orthogonal matrix \( \mathbf{U} = [u_1 u_2 u_3] \) is determined by using
\[ \mathbf{u}_1 = \mathbf{Av}_1 / \| \mathbf{Av}_1 \| = \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix} \]

and then computing an orthonormal basis for \( N(\mathbf{A}^T) \), which is \( \mathbf{u}_2 = \mathbf{e}_2 \) and \( \mathbf{u}_3 = \begin{pmatrix} \sqrt{2} \\ 0 \\ \sqrt{2} \end{pmatrix} \). Therefore, the SVD becomes \( \mathbf{A} = \begin{pmatrix} 1/\sqrt{2} & 0 & \sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & \sqrt{2} \end{pmatrix} \) [1].

4. \( \mathbf{P}_M = \mathbf{uu}^T / (\mathbf{u}^T \mathbf{u}) = (1/10) \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix} \), and \( \mathbf{P}_{M^\perp} = \mathbf{I} - \mathbf{P}_M = (1/10) \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix} \), so \( \mathbf{P}_M \mathbf{b} = \begin{pmatrix} 6 \\ 2 \end{pmatrix} \), and \( \mathbf{P}_{M^\perp} \mathbf{b} = \begin{pmatrix} -2 \\ 6 \end{pmatrix} \).

5. If \( \{\lambda_1, \lambda_2, \ldots, \lambda_n\} \) are the eigenvalues of \( \mathbf{A}_{n \times n} \), then it follows by the diagonalization theorem that

\[
\mathbf{A} = \mathbf{PDP}^{-1} \text{ with } \mathbf{D} = \begin{pmatrix} \lambda_1 \mathbf{I} & 0 & \cdots & 0 \\ 0 & \lambda_2 \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \mathbf{I} \end{pmatrix},
\]

and \( \mathbf{e}^\mathbf{A} = \mathbf{P} \begin{pmatrix} e^{\lambda_1} \mathbf{I} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \mathbf{I} \end{pmatrix} \mathbf{P}^{-1} \). Consequently, \( \{e^{\lambda_1}, e^{\lambda_2}, \ldots, e^{\lambda_n}\} \) are the eigenvalues of \( \mathbf{e}^\mathbf{A} \).

The trace is the sum of the eigenvalues, and the determinant is the product of the eigenvalues, so

\[
\det(\mathbf{e}^\mathbf{A}) = e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = e^{\text{trace}(\mathbf{A})}.
\]

6. The spectral theorem ensures that

\[
\left( \frac{\mathbf{A}}{\lambda_1} \right)^n = \mathbf{G}_1 + \left( \frac{\lambda_2}{\lambda_1} \right)^n \mathbf{G}_2 + \cdots + \left( \frac{\lambda_k}{\lambda_1} \right)^n \mathbf{G}_k \rightarrow \mathbf{G}_1,
\]

and \( \text{rank}(\mathbf{G}_1) = \dim R(\mathbf{G}_1) = \dim N(\mathbf{A} - \lambda_1 \mathbf{I}) = \text{geo mult}_{\mathbf{A}}(\lambda_1) = \text{alg mult}_{\mathbf{A}}(\lambda_1) \) (because \( \mathbf{A} \) is diagonalizable).