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Continuity of the Perron Root

Carl D. Meyer*

Department of Mathematics, North Carolina State University, Raleigh, NC 27695
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That the Perron root of a square nonnegative matrix **A** varies continuously with the entries in **A** is a corollary of theorems regarding continuity of eigenvalues or roots of polynomial equations, the proofs of which necessarily involve complex numbers. But since continuity of the Perron root is a question that is entirely in the field of real numbers, it seems reasonable that there should exist a development involving only real analysis. This article presents a simple and completely self-contained development that depends only on real numbers and first principles.

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1. Introduction

The spectral radius $r = \rho(\mathbf{A})$ of a square matrix with nonnegative entries is called the *Perron root* of **A** because the celebrated Perron–Frobenius theory (summarized below in §2) guarantees that r is an eigenvalue for **A**. If $\{\mathbf{A}_k\}_{k=1}^{\infty}$ is a sequence of $n \times n$ nonnegative matrices with respective Perron roots r_k, and if $\lim_{k\to\infty} \mathbf{A}_k = \mathbf{A}$, then it seems rather intuitive that $\lim_{k\to\infty} r_k = r$, for otherwise something would be dreadfully wrong. But this is not a proof. In fact, a simple self-contained proof depending only on first principles that are strictly in the realm of real numbers seems to have been elusive.

The standard treatment is usually to pawn off the result as a corollary to theorems regarding the continuity of eigenvalues for general matrices. For example, citing the continuity of roots of polynomial equations is an easy dodge, but it fails to satisfy because it buries the issue under complex analysis involving Rouché's theorem which itself requires the argument principle. And then there is Kato's development [1] of eigenvalue continuity built around resolvent integrals, which also requires some heavy lifting with complex analysis. The approach to continuity in [2] that utilizes Schur's decomposition in terms of unitary matrices is concise and can be cited or adapted, but it too must necessarily venture outside the realm of real numbers because there is no real version of Schur's theorem that does the job.

While all reference to complex numbers cannot be completely expunged (e.g., nonnegative matrices can certainly have complex eigenvalues, and the definition of spectral radius given in PF1 below is dependent on them), it is nevertheless true that the continuity of the Perron root is an issue that is entirely in the realm of

*Corresponding author. Email: meyer@ncsu.edu

real numbers, so it seems only reasonable that there should be a simple argument involving only real analysis. The purpose of this article is to present a simple and completely self-contained development that is strictly in the real domain and depends only on rudimentary principles from real analysis together with basic Perron–Frobenius facts as summarized below.

2. Perron–Frobenius Basics

The only Perron–Frobenius facts required to establish the continuity of the Perron root are given here. Details and the complete theory can be found in [3, Chapter 8]. If $\mathbf{A}_{n\times n} \geq \mathbf{0}$ (entrywise) whose spectrum is $\sigma(\mathbf{A})$, then:

PF1. The spectral radius $r = \rho(\mathbf{A}) = \max\left\{ |\lambda| \,\middle|\, \lambda \in \sigma(\mathbf{A}) \right\}$ is an eigenvalue for $\mathbf{A}$.

PF2. There is an associated eigenvector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{Ax} = r\mathbf{x}$, where $\mathbf{x} \geq \mathbf{0}$. Such vectors can always be normalized so that $\|\mathbf{x}\|_1 = 1$, and when this is done, the resulting eigenvector is referred to as a *Perron vector* for $\mathbf{A}$.

PF3. If $\mathbf{A}$ is irreducible (i.e., no permutation similarity transformation of $\mathbf{A}$ can produce a block triangular form with square diagonal blocks), then $r > 0$ and $\mathbf{x} > 0$ when $n \geq 2$.

PF4. If $\mathbf{0} \leq \mathbf{A} \leq \mathbf{B}$ (entrywise), then $\rho(\mathbf{A}) \leq \rho(\mathbf{B})$. In particular, if $\mathbf{A}$ is a square submatrix of $\mathbf{B}$, then $\rho(\mathbf{A}) \leq \rho(\mathbf{B})$.

3. The Development

Throughout, let $\{\mathbf{A}_k\}_{k=1}^{\infty}$ be a sequence of $n \times n$ nonnegative matrices with respective Perron roots r_k, and assume that $\lim_{k\to\infty} \mathbf{A}_k = \mathbf{A}$. The aim is to prove that $\lim_{k\to\infty} r_k = r$, where r is the Perron root of $\mathbf{A}$.

Since each $\mathbf{A}_k \geq \mathbf{0}$, it is apparent that $\mathbf{A} \geq \mathbf{0}$, so the argument can be divided into two cases (or theorems) in which (1) $\mathbf{A}$ is nonnegative and irreducible; and (2) $\mathbf{A}$ is nonnegative and reducible.

The Irreducible Case

When $\mathbf{A}$ is irreducible, the proof is essentially a "one-liner."

Theorem 3.1. *If* $\mathbf{A}$ *is irreducible, then* $r_k \to r$*.*

Proof. Let $\mathbf{E}_k = \mathbf{A}_k - \mathbf{A}$, and let $\mathbf{p}_k$ and $\mathbf{q}^T$ be respective right- and left-hand Perron vectors for $\mathbf{A}_k$ and $\mathbf{A}$ with $\|\mathbf{p}_k\|_1 = 1 = \|\mathbf{q}\|_1$. If $q_\star = \min q_i$, and if $\mathbf{e}$ is a vector of ones, then $\mathbf{q} \geq q_\star \mathbf{e}$, and

$$\mathbf{q}^T \mathbf{p}_k \geq q_\star \mathbf{e}^T \mathbf{p}_k = q_\star > 0 \quad \text{for all } k.$$

Using this with the Cauchy–Schwarz inequality and $\|\mathbf{x}\|_2 \leq \|\mathbf{x}\|_1$ for all $\mathbf{x} \in \mathbb{R}^n$

yields

$$
\begin{aligned}
|(r_k - r)\mathbf{q}^T\mathbf{p}_k| &= |\mathbf{q}^T(r_k\mathbf{p}_k) - (r\mathbf{q}^T)\mathbf{p}_k| = |\mathbf{q}^T(\mathbf{A}_k\mathbf{p}_k) - (\mathbf{q}^T\mathbf{A})\mathbf{p}_k| \\
&= |\mathbf{q}^T(\mathbf{A}_k - \mathbf{A})\mathbf{p}_k| = |\mathbf{q}^T\mathbf{E}_k\mathbf{p}_k| \le \|\mathbf{E}_k\|_2 \\
\implies \quad |r_k - r| &\le \frac{\|\mathbf{E}_k\|_2}{\mathbf{q}^T\mathbf{p}_k} \le \frac{\|\mathbf{E}_k\|_2}{q_\star} \to 0 \quad \implies \quad r_k \to r. \qquad \square
\end{aligned}
$$

The Reducible Case

When $\mathbf{A}$ is reducible, the proof requires a few more lines than the irreducible case.

Theorem 3.2 (The Reducible Case)**.** *If $\mathbf{A}$ is reducible, then $r_k \to r$.*

Proof. If $r = 0$, then $\mathbf{A}$ is nilpotent, say $\mathbf{A}^p = \mathbf{0}$, so

$$
\big[r_k\big]^p = \big[\rho(\mathbf{A}_k)\big]^p = \rho(\mathbf{A}_k^p) \le \big\|\mathbf{A}_k^p\big\| \to \|\mathbf{A}^p\| = 0 \quad \implies \quad r_k \to 0 = r.
$$

Now assume that $r > 0$. The foundation for the remaining part of the proof rests on the following realization.

$$
\left\{ \begin{array}{c} \textit{Every subsequence } \{r_{k_i}\} \textit{ of } \{r_k\} \textit{ has a sub-subsequence } \{r_{k_{i_j}}\} \textit{ such that} \\ r_{k_{i_j}} \to r. \end{array} \right\} \tag{1}
$$

To establish this, adopt the notation $\mathbf{X} \sim \mathbf{Y}$ to mean that $\mathbf{Y} = \mathbf{P}^T\mathbf{X}\mathbf{P}$ for some permutation matrix $\mathbf{P}$ so that $\mathbf{A} \sim \begin{pmatrix} \mathbf{U} & \mathbf{V} \\ \mathbf{0} & \mathbf{W} \end{pmatrix}$, where $\mathbf{U}$ and $\mathbf{W}$ are square. If either $\mathbf{U}$ or $\mathbf{W}$ is reducible, then they in turn can be reduced in the same fashion. Reduction of diagonal blocks can continue until at some point

$$
\mathbf{A} \sim \begin{pmatrix} \bullet & \cdots & \bullet & \cdots & \bullet \\ \vdots & \ddots & \vdots & & \vdots \\ \mathbf{0} & \cdots & \mathbf{B} & \cdots & \bullet \\ \vdots & & \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{0} & \cdots & \bullet \end{pmatrix} \tag{2}
$$

is block triangular with square diagonal blocks, one of which—call it $\mathbf{B}$—is necessarily irreducible and has $\rho(\mathbf{B}) = \rho(\mathbf{A}) = r > 0$. Apply the same symmetric permutation that produced (2) to each $\mathbf{A}_k$ so that

$$
\mathbf{A}_k \sim \begin{pmatrix} \bullet & \cdots & \bullet & \cdots & \bullet \\ \vdots & \ddots & \vdots & & \vdots \\ \bullet & \cdots & \mathbf{B}_k & \cdots & \bullet \\ \vdots & & \vdots & \ddots & \vdots \\ \bullet & \cdots & \bullet & \cdots & \bullet \end{pmatrix},
$$

where $\mathbf{B}_k$ and $\mathbf{B}$ have the same size and occupy the same positions. It follows from (PF4) that if $b_k = \rho(\mathbf{B}_k)$, then $b_k \le r_k$ for each k. And since $\mathbf{A}_k \to \mathbf{A}$ implies $\mathbf{B}_k \to \mathbf{B}$, Case (1) (the irreducible case) ensures $\rho(\mathbf{B}_k) \to \rho(\mathbf{B})$ so that $b_k \to r$. In particular, if $\{r_{k_i}\}$ is any subsequence of $\{r_k\}$, then

$$
b_{k_i} \le r_{k_i} \text{ for each } k_i, \qquad \text{and} \qquad b_{k_i} \to r. \tag{3}
$$

Every subsequence $\{r_{k_i}\}$ is bounded because $\rho(\star) \leq \|\star\|$ for any matrix norm, and this implies that $0 \leq r_{k_i} \leq \|\mathbf{A}_{k_i}\| = \|\mathbf{A} + \mathbf{E}_{k_i}\| \to \|\mathbf{A}\|$. Hence every subsequence $\{r_{k_i}\}$ has a convergent sub-subsequence $r_{k_{i_j}} \to r^\star$. This together with (3) yields

$$b_{k_{i_j}} \leq r_{k_{i_j}} \quad \text{so that} \quad r \leq r^\star. \tag{4}$$

To see that $r^\star = r$, note that the sequence of Perron vectors $\{\mathbf{v}_{k_i}\}$ for $\mathbf{A}_{k_i}$ is bounded because each has norm one, so $\{\mathbf{v}_{k_i}\}$ has a convergent subsequence $\mathbf{v}_{k_{i_j}} \to \mathbf{v}^\star \neq \mathbf{0}$. Use this together with $r_{k_{i_j}} \to r^\star$ to conclude that

$$\mathbf{A}\mathbf{v}^\star = \lim \mathbf{A}_{k_{i_j}} \lim \mathbf{v}_{k_{i_j}} = \lim[\mathbf{A}_{k_{i_j}}\mathbf{v}_{k_{i_j}}] = \lim[r_{k_{i_j}}\mathbf{v}_{k_{i_j}}] = \lim r_{k_{i_j}} \lim \mathbf{v}_{k_{i_j}} = r^\star \mathbf{v}^\star$$
$$\Longrightarrow \quad r^\star \text{ is an eigenvalue for } \mathbf{A} \quad \Longrightarrow \quad r^\star \leq r.$$

This together with (4) ensures that $r^\star = r$, and thus (1) is established. To prove that $r_k \to r$, suppose to the contrary that $r_k \not\to r$ so that there is a subsequence $\{r_{k_s}\}$ and a number $\epsilon > 0$ such that $|r_{k_s} - r| > \epsilon$ for all $s = 1, 2, 3, \ldots$. However, (1) guarantees that $\{r_{k_s}\}$ has a subsequence $\{r_{k_{s_j}}\}$ such that $r_{k_{s_j}} \to r$, which is a contradiction, and thus $r_k \to r$. □

4. A Temptation to Avoid

It is tempting to establish the continuity of the Perron root (or the spectral radius in general) by using the characterization

$$\lim_{m\to\infty} \|\mathbf{X}^m\|^{1/m} = \rho(\mathbf{X}) \tag{5}$$

to simply conclude that

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\lim_{m\to\infty} \|\mathbf{A}_k^m\|^{1/m} = \lim_{m\to\infty}\lim_{k\to\infty} \|\mathbf{A}_k^m\|^{1/m} = \lim_{m\to\infty} \|\mathbf{A}^m\|^{1/m} = r.$$

It would be acceptable to interchange the limits on k and m if the convergence in (5) was uniform on the set $\mathcal{N} = \{\mathbf{X} \in \mathbb{R}^{n\times n} \,|\, \mathbf{X} \geq \mathbf{0}\}$, but alas, it is not. To see this, observe that if $f_m(\mathbf{X}) = \|\mathbf{X}^m\|^{1/m}$ and $f(\mathbf{X}) = \rho(\mathbf{X})$, then $f_m(\alpha\mathbf{X}) = \alpha f_m(\mathbf{X})$ and $f(\alpha\mathbf{X}) = \alpha f(\mathbf{X})$ for all $\alpha \geq 0$ and for all $\mathbf{X} \in \mathcal{N}$. The convergence of f_m to f cannot be uniform because otherwise, for each $\epsilon > 0$, there would exist an integer M such that $m \geq M$ implies

$$|f_m(\mathbf{X}) - f(\mathbf{X})| < \epsilon \quad \text{for all } \mathbf{X} \in \mathcal{N}.$$

In particular, $m \geq M$ implies that

$$|f_m(\alpha\mathbf{X}) - f(\alpha\mathbf{X})| < \epsilon \quad \text{for all } \alpha \geq 0 \text{ and } \mathbf{X} \in \mathcal{N},$$

or equivalently,

$$\alpha\, |f_m(\mathbf{X}) - f(\mathbf{X})| < \epsilon \quad \text{for all } \alpha \geq 0 \text{ and } \mathbf{X} \in \mathcal{N},$$

which is impossible.

5. Concluding Remarks

The developments given in this article provide a simple proof that is entirely contained in the real domain for establishing the continuity of the Perron root of a nonnegative matrix. However, they do not apply for proving the continuity of the spectral radius in general—for this complex analysis cannot be avoided. Even if the limit of a sequence of general matrices is nonnegative, the spectral radius can correspond to a complex eigenvalue so that the techniques of Theorems 3.1 and 3.2 do not apply—e.g., consider

$$\mathbf{A}_k = \begin{pmatrix} 0 & 1 & 0 \\ -1/k & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{6}$$

Even though the continuity of the Perron root is not new, there is nevertheless current interest in extensions such as those given in [4].

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