# In Search of a Matrix


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# Outline

- Background & History
- Vector Space Approach
- Frequency Weighting
- Compression & Noise Reduction
- Examples
- LSI with SVD
- LSI with URV & Centroids
- References

# Background

**Goal:** Identify documents that best match users query

**Measures**

- Recall $= \frac{\#relevant\ docs\ retrieved}{\#docs\ in\ collection}$ (max # useful docs)
- Precision $= \frac{\#relevant\ docs\ retrieved}{\#docs\ retrieved}$ (min # useless docs)
- Do it $FAST!$

**Methods**

- Combinatorial
- Statistical
- Hashing
- Pattern matching
- Vector Space

# SMART

(**S**ystem for the **M**echanical **A**nalysis and **R**etrieval of **T**ext)

## Harvard 1962 – 1965

- IBM 7094 & IBM 360

## Gerard Salton

- Implemented at Cornell (1965 – 1970)
- Based on matrix methods

# Term–Document Matrix

## Start With Dictionary of Terms

- Single words — or short phrases (e.g., *landing gear*)

## Index Each Document (by human or by computer)

- Count $f_{ij}$ = # times term $i$ appears in document $j$

## Unweighted Term–Document Matrix

$$\begin{array}{c c} & \begin{array}{cccc} \textsc{Doc } 1 & \textsc{Doc } 2 & \cdots & \textsc{Doc } n \end{array} \\ \begin{array}{c} \textsc{Term } 1 \\ \textsc{Term } 2 \\ \vdots \\ \textsc{Term } m \end{array} & \begin{pmatrix} f_{\mathbf{11}} & f_{\mathbf{12}} & \cdots & f_{\mathbf{1}n} \\ f_{\mathbf{21}} & f_{\mathbf{22}} & \cdots & f_{\mathbf{2}n} \\ \vdots & \vdots & \ddots & \vdots \\ f_{m\mathbf{1}} & f_{m\mathbf{2}} & \cdots & f_{mn} \end{pmatrix} \end{array} = \mathbf{A}_{m\times n}$$

## Features

- $\mathbf{A} \geq \mathbf{0}$
- **A** can be really big! — Terms = $O(\mathbf{10^6})$ Docs = $O(\mathbf{10^7})$
- **A** is sparse — but otherwise unstructured
- **A** contains a lot of uncertainty (noise)

# Example

(M. W. Berry & M. Browne, 1999, SIAM)

| Terms | | Documents | |
|---|---|---|---|
| T1: | Bab(y, ies, y's) | D1: | Infant and toddler first aid |
| T2: | Child(ren's) | D2: | Babies and children's room for your home |
| T3: | Guide | D3: | Child safety at home |
| T4: | Health | D4: | Your baby's health and safety: from infant to toddler |
| T5: | Home | D5: | Baby proofing basics |
| T6: | Infant | D6: | Your guide to easy rust proofing |
| T7: | Proofing | D7: | Beanie babies collector's guide |
| T8: | Safety | | |
| T9: | Toddler | | |

$$
\mathbf{A} = \begin{array}{c} \\ T1 \\ T2 \\ T3 \\ T4 \\ T5 \\ T6 \\ T7 \\ T8 \\ T9 \end{array}
\begin{array}{c} \begin{array}{ccccccc} D1 & D2 & D3 & D4 & D5 & D6 & D7 \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}_{9\times 7} \end{array}
$$

# Example

(M. W. Berry & M. Browne, 1999, SIAM)

| Terms | Documents |
|---|---|
| T1: Bab(y, ies, y's) | D1: Infant and toddler first aid |
| T2: Child(ren's) | D2: Babies and children's room for your home |
| T3: Guide | D3: Child safety at home |
| T4: Health | D4: Your baby's health and safety: from infant to toddler |
| T5: Home | D5: Baby proofing basics |
| T6: Infant | D6: Your guide to easy rust proofing |
| T7: Proofing | D7: Beanie babies collector's guide |
| T8: Safety | |
| T9: Toddler | |

$$
\mathbf{A} = \begin{array}{c} \\ \mathrm{T1} \\ \mathrm{T2} \\ \mathrm{T3} \\ \mathrm{T4} \\ \mathrm{T5} \\ \mathrm{T6} \\ \mathrm{T7} \\ \mathrm{T8} \\ \mathrm{T9} \end{array}
\begin{array}{c}
\begin{array}{ccccccc} \mathrm{D1} & \mathrm{D2} & \mathrm{D3} & \mathrm{D4} & \mathrm{D5} & \mathrm{D6} & \mathrm{D7} \end{array} \\
\begin{pmatrix}
0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}_{9 \times 7}
\end{array}
$$

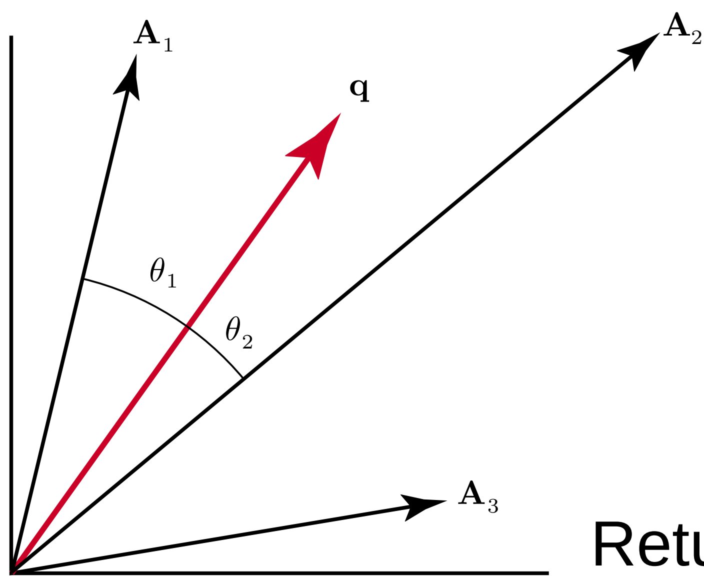
# Query Matching

## Query Vector

- $\mathbf{q}^T = (q_1, q_2, \ldots, q_m)$ where $q_i = \begin{cases} 1 & \text{if Term } i \text{ is requested} \\ 0 & \text{if not} \end{cases}$

## How Close is the Query to Each Document?

- i.e., how close is $\mathbf{q}$ to each column $\mathbf{A}_k$?




$\|\mathbf{q} - \mathbf{A}_1\| < \|\mathbf{q} - \mathbf{A}_2\|$ but $\theta_2 < \theta_1$

Use $\delta_i = \cos\theta_i = \dfrac{\mathbf{q}^T \mathbf{A}_i}{\|\mathbf{q}\| \, \|\mathbf{A}_i\|}$

Rank documents by size of $\delta_i$

Return Document $i$ to user when $\delta_i \geq tol$

# Term Weighting

## A Defect

- If the term $bank$ occurs once in Doc $\mathbf{1}$ but twice in Doc $\mathbf{2}$, and if $\|\mathbf{A_1}\| \approx \|\mathbf{A_2}\|$, then a query containing only $bank$ produces $\delta_\mathbf{2} \approx \mathbf{2}\delta_\mathbf{1}$ (i.e., Doc $\mathbf{2}$ is rated twice as relevant as Doc $\mathbf{1}$).

## To Compensate

- Set $a_{ij} = \log(\mathbf{1} + f_{ij})$ (other weights also possible)

## Query Weights

- Terms $Boeing$ and $airplanes$ not equally important in query
- Importance of Term $i$ tends to be inversely proportional to $\nu_i$ = # Docs containing Term $i$

## To Compensate

- Set $q_i = \begin{cases} \log(n/\nu_i) & \text{if } \nu_i \neq \mathbf{0} \\ \mathbf{0} & \text{if } \nu_i = \mathbf{0} \end{cases}$ (other weights also possible)

# Noise in A

## Ambiguity in Vocabulary

- e.g., A $bank$ could be
  - A financial institution
  - A river side
  - A shot in the game of pool

## Variation in Writing Style

- No two authors write the same way
  - One author may write $car$ and $laptop$
  - Another author may write $automobile$ and $portable$

## Variation in Indexing Conventions

- No two people index documents the same way
- Computer indexing is inexact and can be unpredictable

# Practical Problems

## Simple in Theory

- Weight terms and normalize cols — Make $\|\mathbf{A}_i\| = 1$
- For each new query, weight and normalize — Make $\|\mathbf{q}\| = 1$
- Compute $\delta_i = \cos\theta_i = (\mathbf{q}^T\mathbf{A})_i$ and return most relevant docs

## Difficult in Practice

- Must be able to do it $FAST!$

  $\Longrightarrow$ Somehow compress the data

- Must account for variations due to ambiguity in language and variations in writing and indexing styles.

  $\Longrightarrow$ Somehow reduce "noise" or "uncertainty" in **A**

# Basis Games

## Consider Vector of Data

$\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{e}_i$ with respect to o.n. basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$

$\Longrightarrow \alpha_i = \langle \mathbf{e}_i | \mathbf{v} \rangle$ = amount of $\mathbf{v}$ in direction of $\mathbf{e}_i$

## Compression

- Select new o.n. basis $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n\}$ so that fewer vectors are needed to represent $\mathbf{v} = \sum_{i=1}^{\mathbf{r}} \beta_i \mathbf{u}_i$ $\quad (\mathbf{r} < n)$
- Data is compressed from $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ into $\{\beta_1, \beta_2, \cdots, \beta_{\mathbf{r}}\}$

## Even More Compression

- Eliminate data lying in insignificant directions

  Arrange: $|\beta_1| \geq |\beta_2| \geq \cdots \geq |\beta_{\mathbf{k}}| \geq \epsilon > |\beta_{\mathbf{k}+1}| \cdots \geq |\beta_r|$

  Approximate: $\mathbf{v} \approx \widetilde{\mathbf{v}} = \sum_{i=1}^{\mathbf{k}} \beta_i \mathbf{u}_i$ $\quad (\mathbf{k} < \mathbf{r} < n)$
- Data now compressed from $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ to $\{\beta_1, \beta_2, \cdots, \beta_{\mathbf{k}}\}$

# Added Benefit

## Noise Reduction

- Assume noise (or uncertainty) is nondirectional

  $\Longrightarrow$ As much noise in one direction as in any other direction

  $\Longrightarrow$ $\mathbf{v} = \sum_{i=1}^{r} \beta_i \mathbf{u}_i = \sum_{i=1}^{r} s_i \mathbf{u}_i + \sum_{i=1}^{r} \epsilon \mathbf{u}_i$ = (signal) + (noise)

- Suppose $|s_1| \geq |s_2| \geq \cdots \geq |s_k| \geq \epsilon > |s_{k+1}| \cdots \geq |s_r|$

  Drop $\beta_{k+1}, \beta_{k+2}, \cdots, \beta_r$, and use $\mathbf{v} \approx \tilde{\mathbf{v}} = \sum_{i=1}^{k} \beta_i \mathbf{u}_i$

  $\Longrightarrow$ Only a small proportion of the signal is lost

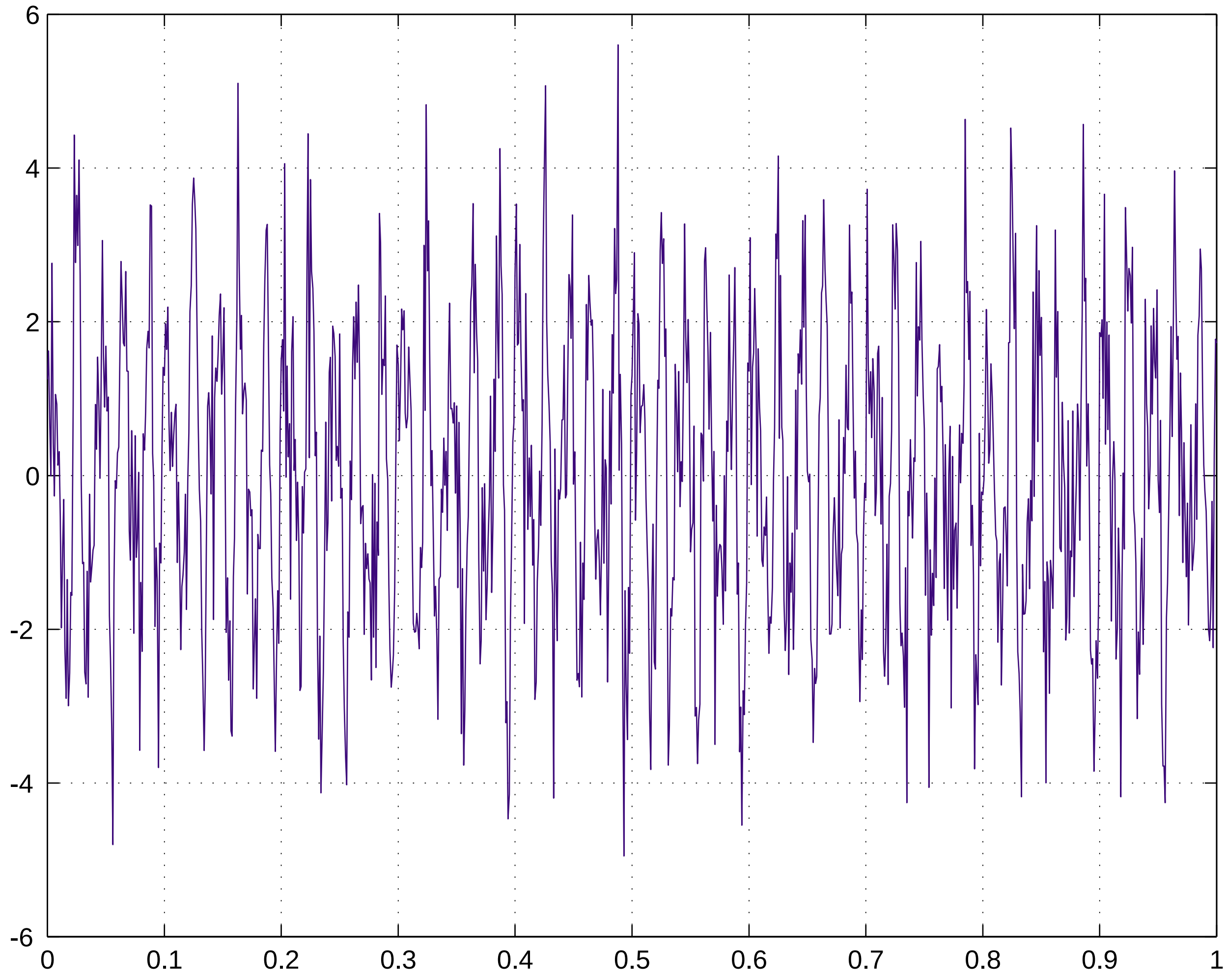
  $\Longrightarrow$ A larger proportion of the noise is lost

# Example

(Matrix Analysis and Applied Linear Algebra, SIAM, 2000)

## Sample Audio Signal 512 times for 1 sec.

$$\mathbf{x} = \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{510} \\ x_{511} \end{bmatrix}$$




## Goal: Compress and Reduce Noise

# Find A Better Basis

## Oscillatory $\Longrightarrow$ Cosines & Sines

$$\mathbf{x}(t) \sim \sum_{f=1}^{\infty} \alpha_f \cos 2\pi f t + \beta_f \sin 2\pi f t \qquad (\text{if } \mathbf{x}(t) \text{ was a continuous function})$$

## Discrete Time, Cosine, and Sine Vectors

$$\mathbf{t} = \begin{bmatrix} {}^{0}/_{n} \\ {}^{1}/_{n} \\ {}^{2}/_{n} \\ \vdots \\ {}^{n-1}/_{n} \end{bmatrix} \quad \cos 2\pi f \mathbf{t} = \begin{bmatrix} \cos\left(2\pi f \cdot \frac{0}{n}\right) \\ \cos\left(2\pi f \cdot \frac{1}{n}\right) \\ \cos\left(2\pi f \cdot \frac{2}{n}\right) \\ \vdots \\ \cos\left(2\pi f \cdot \frac{n-1}{n}\right) \end{bmatrix} \quad \sin 2\pi f \mathbf{t} = \begin{bmatrix} \sin\left(2\pi f \cdot \frac{0}{n}\right) \\ \sin\left(2\pi f \cdot \frac{1}{n}\right) \\ \sin\left(2\pi f \cdot \frac{2}{n}\right) \\ \vdots \\ \sin\left(2\pi f \cdot \frac{n-1}{n}\right) \end{bmatrix}$$

## Discrete Exponential Vectors

$$e^{i2\pi f \mathbf{t}} = \cos 2\pi f \mathbf{t} + i \sin 2\pi f \mathbf{t} \qquad e^{-i2\pi f \mathbf{t}} = \cos 2\pi f \mathbf{t} - i \sin 2\pi f \mathbf{t}$$

# Discrete Fourier Transform

$$\omega = e^{2\pi i/n} \qquad \mathbf{W} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^{n-1} \\ 1 & \omega^2 & \omega^4 & \cdots & \omega^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{n-1} & \omega^{n-2} & \cdots & \omega \end{bmatrix}_{n\times n} \qquad \mathbf{W}_f = \frac{1}{2}e^{2\pi i f \mathbf{t}}$$

## Identities

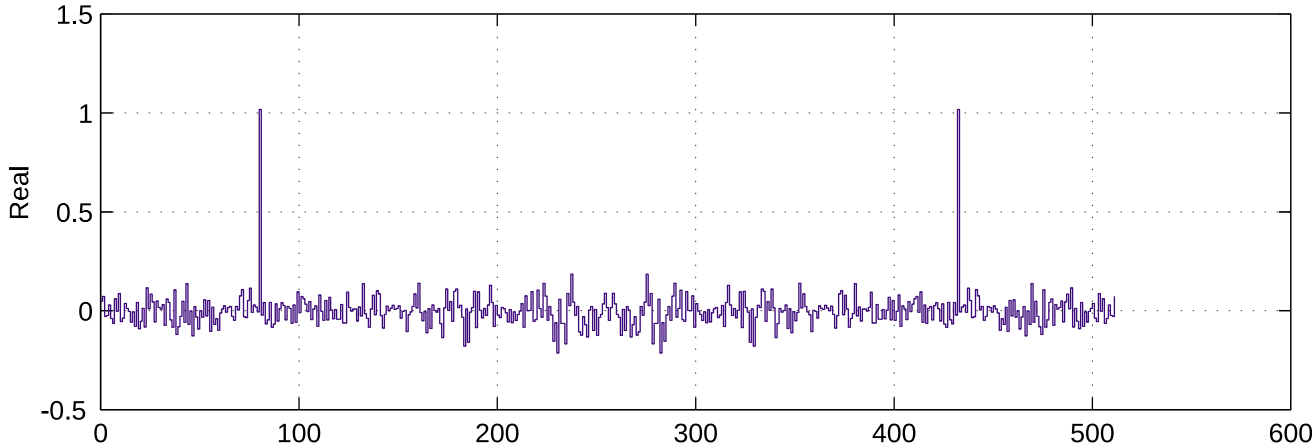
- $\cos 2\pi f\mathbf{t} = \mathbf{W}_f + \mathbf{W}_{n-f}$ $\quad (0 < f < n)$
- $\sin 2\pi f\mathbf{t} = -i(\mathbf{W}_f - \mathbf{W}_{n-f})$

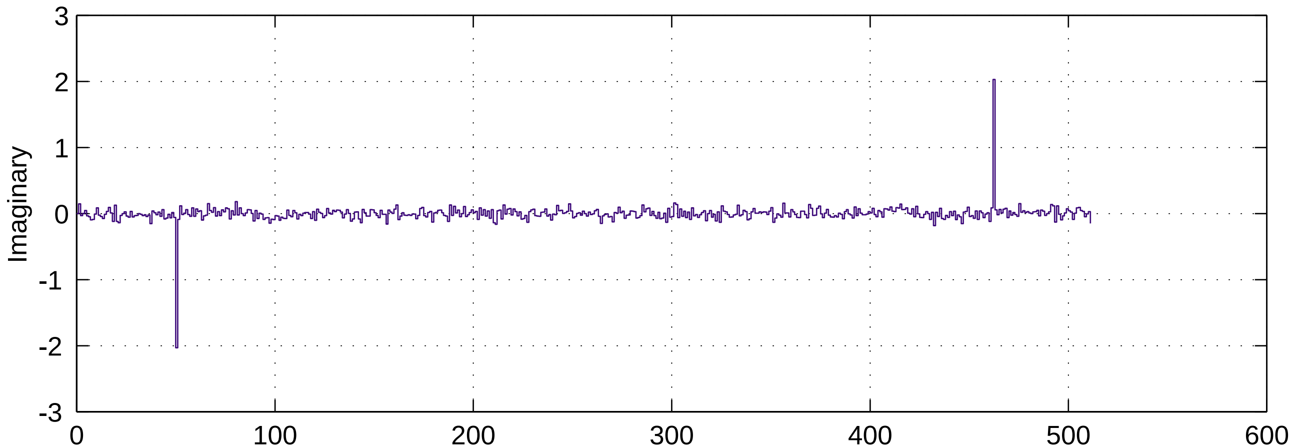
**New Basis** $= \{\mathbf{W}_0, \mathbf{W}_1, \ldots, \mathbf{W}_{n-1}\}$ (Find coordinates of $\mathbf{x}$ wrt $\mathbf{W}_i$'s)

$$\mathbf{x} = \mathbf{W}\mathbf{y} \implies \mathbf{y} = \mathbf{W}^{-1}\mathbf{x} = \frac{2}{n}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{n-2} & \cdots & \xi \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix}$$

$$\xi = e^{-2\pi i/n} = \overline{\omega}$$

# The New Coordinates

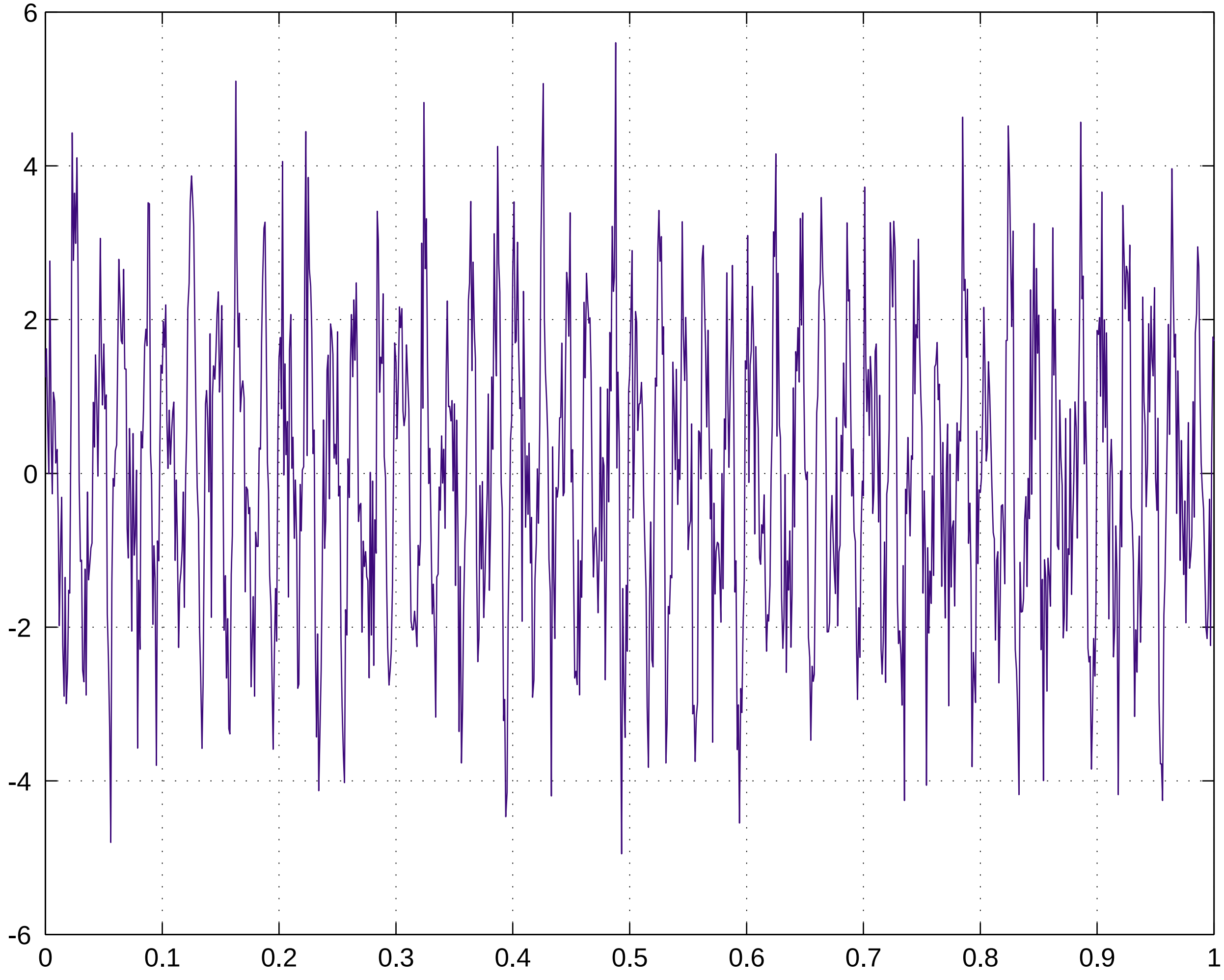


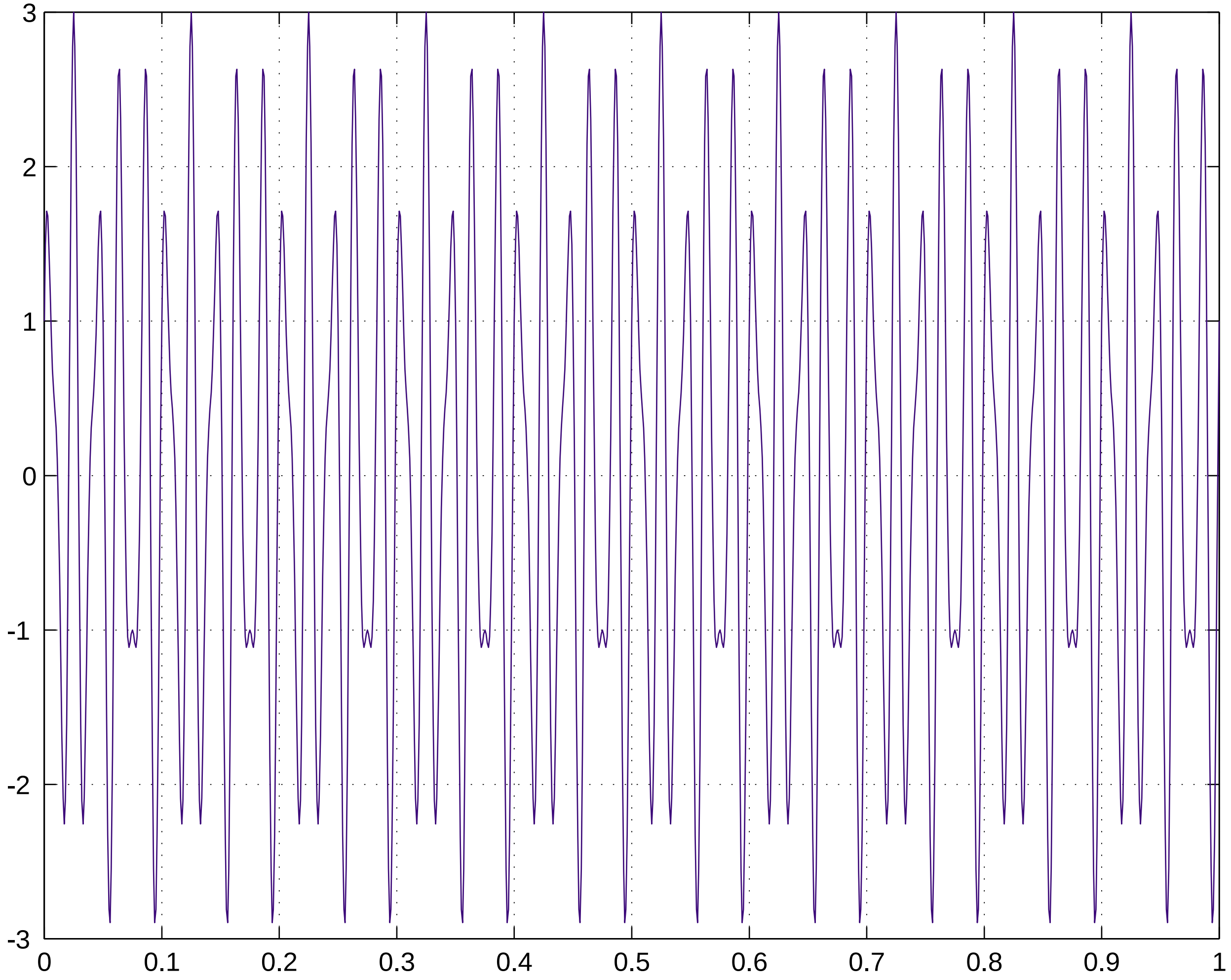

$$\mathbf{y} = (\mathbf{e}_{80} + \mathbf{e}_{432}) - 2\mathrm{i}(\mathbf{e}_{50} - \mathbf{e}_{462}) + \varepsilon$$
$$= (\mathbf{e}_{80} + \mathbf{e}_{n-80}) - 2\mathrm{i}(\mathbf{e}_{50} - \mathbf{e}_{n-50}) + \varepsilon \qquad (n = 512)$$

$$\mathbf{x} = \mathbf{W}\mathbf{y} = (\mathbf{W}_{80} + \mathbf{W}_{n-80}) - 2\mathrm{i}(\mathbf{W}_{50} - \mathbf{W}_{n-50}) + \varepsilon$$
$$= \cos 2\pi 80\mathbf{t} + 2\sin 2\pi 50\mathbf{t} + \varepsilon$$

# Original



# Cleaned & Compressed




$$\cos 2\pi 80t + 2 \sin 2\pi 50t$$

# The DFT Game

## Matrix–Vector Product

$$\mathbf{y} = \frac{2}{n}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \xi & \xi^2 & \cdots & \xi^{n-1} \\ 1 & \xi^2 & \xi^4 & \cdots & \xi^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \xi^{n-1} & \xi^{n-2} & \cdots & \xi \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_{n-1} \end{bmatrix} \qquad \xi = \mathrm{e}^{-2\pi \mathrm{i}/n}$$

## Simple in Theory but Difficult in Practice

- Must do it *FAST!*

## Need For Speed $\Longrightarrow$ Matrix Factorizations $\Longrightarrow$ FFT

- $\mathbf{F}_n = \mathbf{B}_n(\mathbf{I}_2 \otimes \mathbf{F}_{n/2})\mathbf{P}_n \qquad \mathbf{B}_n = \begin{bmatrix} \mathbf{I}_{n/2} & \mathbf{D}_{n/2} \\ \mathbf{I}_{n/2} & -\mathbf{D}_{n/2} \end{bmatrix} \qquad \mathbf{D}_{n/2} = \begin{bmatrix} 1 & & & \\ & \xi & & \\ & & \xi^2 & \\ & & & \ddots \end{bmatrix}$
- FFT changes $n^2$ flop requirement into $(n/2)\log_2 n$

"The most valuable numerical algorithm in our lifetime."

—G. Strang, Bulletin of the AMS, April, 1993.

# Things Have Changed

- “For engineers and social and physical scientists, linear algebra now fills a place that is often more important than calculus.”
- “It is partly [due to] the move from analog to digital; functions are replaced by vectors.”
- “My generation of students, and certainly my teachers, did not see this change coming.”

—Gilbert Strang

FROM THE AMERICAN SCIENTIST

APRIL, 1994.

# Back To IR

## Almost the Same Problem

- Evaluate $\mathbf{q}^T\mathbf{A}$ fast

## Data is Now the Term-Doc Matrix in Standard Coordinates

- $\mathbf{A} = \sum_{i,j} \langle \mathbf{E}_{ij} | \mathbf{A} \rangle \, \mathbf{E}_{ij}$ $\qquad \mathbf{E}_{ij} = \mathbf{e}_i \mathbf{e}_j^T$ $\qquad \langle \mathbf{E}_{ij} | \mathbf{A} \rangle = a_{ij}$
- $\langle \mathbf{X} | \mathbf{Y} \rangle = trace\left(\mathbf{X}^{\mathsf{T}}\mathbf{Y}\right)$ $\qquad \|\mathbf{X}\| = \langle \mathbf{X} | \mathbf{X} \rangle^{1/2}$ (Frobenius Norm)

## Seek New o.n. Basis That Squeezes & Cleans

- $\mathbf{A} = \sum_{i=1}^{r} \langle \mathbf{Z}_i | \mathbf{A} \rangle \, \mathbf{Z}_i$

## Think Matrix Factorizations $\Longrightarrow$ SVD $\Longrightarrow$ $\mathbf{A} = \mathbf{UDV}^T$

- $\mathbf{A} = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \sum_{i=1}^{r} \sigma_i \mathbf{Z}_i, \qquad \mathbf{Z}_i = \mathbf{u}_i \mathbf{v}_i^T, \qquad \langle \mathbf{Z}_i | \mathbf{Z}_j \rangle = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$
- $\sigma_i = \langle \mathbf{Z}_i | \mathbf{A} \rangle$ = the amount of $\mathbf{A}$ in direction of $\mathbf{Z}_i$

# Same As Before

## Assume Nondirectional Noise or Uncertainty

- Drop small $\sigma_i$'s and use $\mathbf{A} \approx \widetilde{\mathbf{A}} = \sum_{i=1}^{k} \sigma_i \mathbf{Z}_i$
- Lose only small part of relevance
- Lose larger proportion of noise or uncertainty

## Be Liberal in Dropping $\sigma_i$'s

- Numerical accuracy not important — 2 or 3 significant digits

## New Query Matching Strategy

- Normalize
  - $\mathbf{q} \leftarrow \mathbf{q}/\|\mathbf{q}\|$
  - $\widetilde{\mathbf{A}} \leftarrow \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T \mathbf{D} = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \widetilde{\mathbf{v}}_i^T$
- Compare query to each document
  - $(\delta_1, \delta_2, \ldots, \delta_n) = \mathbf{q}^T \widetilde{\mathbf{A}} = \sum_{i=1}^{k} \sigma_i (\mathbf{q}^T \mathbf{u}_i) \widetilde{\mathbf{v}}_i^T$

# Pros & Cons

## Advantages

- Compression
  - **A** is replaced by only a few sing values and sing vectors
  - They are determined & normalized only once
- $SPEED!$
  - Each query requires only a few inner products

$$\mathbf{q}^T\widetilde{\mathbf{A}}_{m\times n} = \sum_{i=1}^{k} \sigma_i(\mathbf{q}^T\mathbf{u}_i)\widetilde{\mathbf{v}}_i^T$$

- Latent semantic associations are made
  - Relevant docs not found by direct matching show up

## Disadvantages

- Adding & deleting docs (updating & downdating SVD) difficult
- Determining optimal $k$ is not easy (empirical tuning required)

# Variations

## Projected Query

- First project the query onto the document space

$$\widetilde{\mathbf{q}} = \mathbf{P}_{R(\mathbf{A})}\mathbf{q} = \sum_{j=1}^{r} \mathbf{u}_j \mathbf{u}_j^T \mathbf{q}$$

- Or, better yet, use truncated projection

$$\widetilde{\mathbf{q}} = \mathbf{P}_{R\left(\widetilde{\mathbf{A}}\right)}\mathbf{q} = \sum_{j=1}^{k} \mathbf{u}_j \mathbf{u}_j^T \mathbf{q}$$

- Notice
  - $\widetilde{\mathbf{q}}^T \widetilde{\mathbf{A}} = \sum_{i=1}^{k} \sigma_i (\widetilde{\mathbf{q}}^T \mathbf{u}_i) \widetilde{\mathbf{v}}_i^T = \sum_{i=1}^{k} \sigma_i \left( (\sum_{j=1}^{k} \mathbf{q}^T \mathbf{u}_j^T \mathbf{u}_j) \mathbf{u}_i \right) \widetilde{\mathbf{v}}_i^T = \mathbf{q}^T \widetilde{\mathbf{A}}$
  - $\|\widetilde{\mathbf{q}}\| \leq \|\mathbf{q}\|$
  - $\cos \widetilde{\theta}_i \geq \cos \theta_i$ (more documents are deemed relevant)

# Other Factorizations

## DFT — FFT

- No compression — no oscillatory components

## Haar Transform $\mathbf{H}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad \mathbf{H}_4 = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}$

- $\mathbf{H}_n = (\mathbf{I}_2 \otimes \mathbf{H}_{n/2})\mathbf{P}_n \begin{bmatrix} \mathbf{H}_{n/2} & \\ & \mathbf{I}_{n/2} \end{bmatrix} \quad \Rightarrow \quad \mathbf{H}_n\mathbf{x}$ is $Fast$ (if $n = 2^p$)
- Factor $\mathbf{A} = \mathbf{H}_m\mathbf{B}\mathbf{H}_n^T = \sum_{i,j} \beta_{ij}\mathbf{h}_i\mathbf{h}_j^T$ ( $\mathbf{h}$'s only use $-1$, $0$, or $1$)
  - More than a few $\beta_{ij}$'s may be needed
  - Needs padding if $m$ or $n$ not a power of 2

## Semidiscrete Decomposition (T. Kolda and D. O'Leary, 1998)

- Approximate $\mathbf{A} \approx \sum_{i=1}^{k} \alpha_i\mathbf{x}_i\mathbf{y}_j$ $\quad \mathbf{x}_i$ and $\mathbf{y}_j$ only use $-1$, $0$, or $1$

## Other Wavelet Transforms?

# Using Centroids

## Document Clusters

- Assume $\mathbf{A} = [\mathbf{a}_1\ \mathbf{a}_2\ \cdots\ \mathbf{a}_n]$ (normalized) represents a cluster
- Mean: $\mathbf{m} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{a}_i = \frac{\mathbf{Ae}}{n}$ where $\mathbf{e}^T = (1, 1, \ldots, 1)$
- Centroid vector: $\mathbf{c} = \mathbf{m}/\|\mathbf{m}\| = \mathbf{Ae}/\|\mathbf{Ae}\|$

## Facts

- $$\min_{\mathbf{p}\geq 0, \|\mathbf{p}\|=1} \sum_{i=1}^{n} \cos\theta(\mathbf{a}_i, \mathbf{p}) = \sum_{i=1}^{n} \cos\theta(\mathbf{a}_i, \mathbf{c}).$$
- i.e., $\mathbf{c}$ is the closest vector to the cluster
- $\mathbf{c}^T\mathbf{A} \approx \mathbf{e}^T \qquad (\mathbf{c}^T\mathbf{a}_i = \cos\theta_i \approx 1)$

## Reflectors With Specified Unit Column — Say $\mathbf{c}$

- $\mathbf{L} = \mathbf{I} - 2\mathbf{u}\mathbf{u}^T/(\mathbf{u}^T\mathbf{u}) = [\mathbf{c} \mid \mathbf{X}]$ where $\mathbf{u} = \mathbf{c} \pm \mathbf{e}_1$
- $\mathbf{L} = \mathbf{L}^T = \mathbf{L}^{-1}$

# Orthogonal Factorization

## Specify $\mathbf{c}$ and $\mathbf{e}/\sqrt{n}$ as First Columns

- $\mathbf{L} = [\mathbf{c} \mid \mathbf{X}]$ and $\mathbf{R} = [\mathbf{e}/\sqrt{n} \mid \mathbf{Y}]$

$$\mathbf{LAR} = \mathbf{L}^T\mathbf{AR} = \begin{bmatrix} \mathbf{c}^T\mathbf{Ae}/\sqrt{n} & \mathbf{c}^T\mathbf{AY} \\ \mathbf{X}^T\mathbf{Ae}/\sqrt{n} & \mathbf{X}^T\mathbf{AY} \end{bmatrix} \approx \begin{bmatrix} \mathbf{e}^T\mathbf{e}/\sqrt{n} & \mathbf{e}^T\mathbf{Y} \\ \mathbf{X}^T\mathbf{c}\|\mathbf{Ae}\|/\sqrt{n} & \mathbf{X}^T\mathbf{AY} \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{n} & 0 \\ 0 & \mathbf{X}^T\mathbf{AY} \end{bmatrix} = \begin{bmatrix} \sqrt{n} & 0 \\ 0 & \mathbf{B} \end{bmatrix}$$

## A New Factorization

- $\mathbf{A} = \mathbf{LDR} = \mathbf{LDR}^T = \mathbf{ce}^T + \sum_{i,j \neq 1} \beta_{ij}\mathbf{x}_i\mathbf{y}_j^T$

## Truncate To Compress & Clean

## Break Entire Collection Into Clusters

# To Learn More

## Books

- Matrix Analysis and Applied Linear Algebra, C. D. Meyer, SIAM, 2000.
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