Applying theory of Markov chains to the problem of ranking

A. Govan C. Meyer

Department of Mathematics
North Carolina State University

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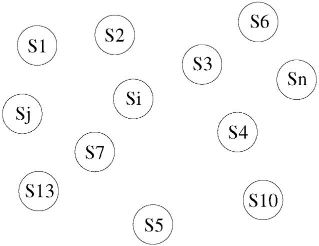
Outline

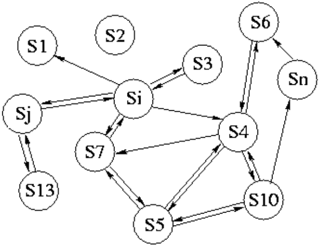
Markov Chains Basics

States-finite:



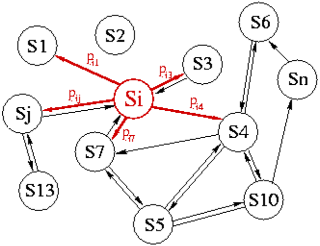
Markov Chains Basics

Transitioning between states:



Markov Chains Basics

Transition probabilities:



Markov Chains Basics-Probability

Restrictions on the transition probabilities:

- Memoryless (Markov property)
  - $p_{i_t j} = P(X_{t+1} = S_j | X_t = S_{i_t}, X_{t-1} = S_{i_{t-1}}, ..., X_0 = S_{j_0}) = P(X_{t+1} = S_j | X_t = S_{i_t})$

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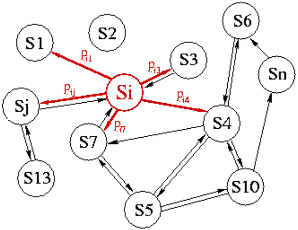
- Homogeneous
  - p_{ij} has no time dependence

Markov Chains Basics-Matrices



Transition probability matrix-stochastic matrix

$$\begin{array}{c} S_1 \\ \vdots \\ S_i \\ \vdots \\ S_n \end{array} \begin{pmatrix} & S_1 & S_2 & \dots & S_i & \dots & S_n \\ & \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ & p_{i1} & p_{i2} & \dots & p_{ii} & \dots & p_{in} \\ & \vdots & \vdots & \dots & \vdots & \dots & \vdots \end{pmatrix}$$

Markov Chains-Stochastic Matrices

- Markov Chain:
  - $\{\mathbf{p}(0), \mathbf{p}(1), \mathbf{p}(2), ...\}$
    such that $\mathbf{p}(i)$ is a probability distribution vector and
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- some Stochastic Matrices are:
  - Irreducible
    - from any state to any state
  - Primitive
    - $\lambda = 1$ is the only one on the spectral circle
    - can use power method to compute stationary distribution vector (eigenvector corresponding to $\lambda = 1$)

Making Google Matrix

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  - $\mathbf{H}(i,j) = \begin{cases} 1/|i| & \text{there is a link from } i \text{ to } j \\ 0 & \text{otherwise} \end{cases}$

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- Google Matrix **G**.
  - Convex combination: $\mathbf{G} = \alpha\mathbf{S} + (1-\alpha)\mathbf{e}\mathbf{v}^T$, $\alpha \in (0,1)$ and $\mathbf{v}^T > 0$
  - Personalization vector **v**.

PageRank vector π.

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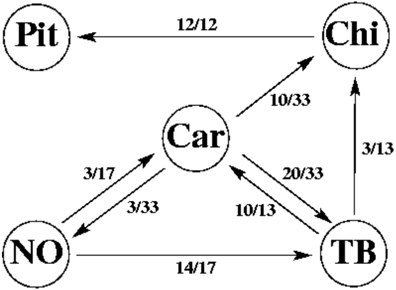
- π is the stationary probability distribution vector.
- π is unique (up to a scalar multiple).

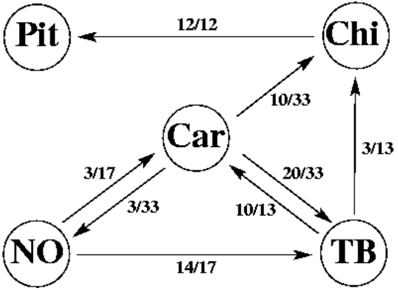
NFL set up.

- Each NFL team is a state.

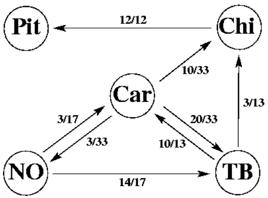


NFL set up.

- Each NFL team is a state.
- Score differences determine transition probability



NFL Matrix



$$
\begin{array}{c|ccccc}
 & \text{Car} & \text{Pit} & \text{Chi} & \text{TB} & \text{NO} \\
\text{Car} & 0 & 0 & \frac{10}{33} & \frac{20}{33} & \frac{3}{33} \\
\text{Pit} & 0 & 0 & 0 & 0 & 0 \\
\text{Chi} & 0 & 1 & 0 & 0 & 0 \\
\text{TB} & \frac{10}{13} & 0 & \frac{3}{13} & 0 & 0 \\
\text{NO} & \frac{3}{17} & 0 & 0 & \frac{14}{17} & 0
\end{array}
$$

NFL Matrix-Stochastic

Dealing with an undefeated team:

	Car	Pit	Chi	TB	NO
Car	0	0	10/33	20/33	3/33
Pit	1/5	1/5	1/5	1/5	1/5
Chi	0	1	0	0	0
TB	10/13	0	3/13	0	0
NO	3/17	0	0	14/17	0

NFL Matrix-Irreducible and Primitive

- Adding stats:

$$\mathbf{F} = \alpha \mathbf{S} + (1 - \alpha)\mathbf{e}\mathbf{s}^T$$

where **s** is based on teams statistical data.

NFL Matrix-Irreducible and Primitive

- Adding stats:
$$\mathbf{F} = \alpha\mathbf{S} + (1-\alpha)\mathbf{es}^T$$
where $\mathbf{s}$ is based on teams statistical data.
- Adding more stats?
$$\mathbf{F} = \alpha_0\mathbf{S} + \alpha_1\mathbf{es}_1^T + ... + \alpha_k\mathbf{es}_k^T$$
where $\mathbf{s}_i$ is statistics based and $\sum \alpha_i = 1$.

Current and Future Work:

- Determining the "important" statistics vectors $\mathbf{s}_1^T, \ldots, \mathbf{s}_k^T$
- Automate the selection of the best α_i for a specified $\mathbf{s}_i^T$.