Updating The PageRank Vector

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The PageRank Vector

Definition

$\pi^T$ = stationary distribution of a Markov chain

$$P = tT + (1 - t)E \quad 0 < t < 1$$

Irreducible & Aperiodic
The PageRank Vector

Definition
\[ \pi^T = \text{stationary distribution of a Markov chain} \]
\[ P = tT + (1 - t)E \quad 0 < t < 1 \]

Irreducible & Aperiodic

Big Eigenvector Problem

Solve \[ \pi^T = \pi^T P \quad \pi^T e = 1 \]

\[ n = O(10^9) \quad \text{(too big for direct solves)} \]

“World’s Largest Matrix Computation”  

(Cleve Moler)
Computing $\pi^T$

Iterate

Start with $\pi_0^T = e/n$ and iterate $\pi_{j+1}^T = \pi_j^T P$ (power method)

Convergence Time

Use to be measured in days
Computing $\pi^T$

Iterate

Start with $\pi_0^T = e/n$ and iterate $\pi_j^T = \pi_j^T P$

(power method)

Convergence Time

Use to be measured in days

Now ???

Recent Advances

Extrapolation methods for accelerating PageRank, Kamvar, Haveliwala, Manning, Golub, 03

Exploiting the block structure of the web for computing PageRank, K, H, M, Golub, 03

Adaptive methods for the computation of PageRank, Kamvar, Haveliwala, Golub, 03

Partial state space aggregation based on lumpability and its application to PageRank, Chris Lee, 03
Updating

Easy Problem

No pages added — No pages removed

— Size does not change — only probabilities change
Updating

**Easy Problem**

No pages added — No pages removed

- Size does not change — only probabilities change

**Hard Problem**

Both pages & links are added or removed

- Both size & probabilities change
Updating

Easy Problem

No pages added — No pages removed

- Size does not change — only probabilities change

Hard Problem

Both pages & links are added or removed

- Both size & probabilities change

The Trouble

Prior results are not much help

- Google just restarts from scratch every few weeks
Perron Complementation

**Perron Frobenius**

\[ P \geq 0 \text{ irreducible} \implies \rho = \rho(P) \text{ simple eigenvalue} \]

**Unique Left-Hand Perron Vector**

\[ \pi^T P = \rho \pi^T \quad \pi^T > 0 \quad \| \pi^T \|_1 = 1 \]
Perron Complementation

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\[ P \geq 0 \quad \text{irreducible} \quad \Rightarrow \quad \rho = \rho(P) \quad \text{simple eigenvalue} \]

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**Partition**

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]

Shift \( P \) by \( \rho \) \quad \rightarrow \quad \text{Schur Complements} \quad \rightarrow \quad \text{Shift back by } \rho
Perron Complementation

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Shift \( P \) by \( \rho \) \quad \rightarrow \quad \text{Schur Complements} \quad \rightarrow \quad \text{Shift back by} \ \rho

**Perron Complements**

\[ S_1 = P_{11} + P_{12}(\rho I - P_{22})^{-1}P_{21} \]

\[ S_2 = P_{22} + P_{21}(\rho I - P_{11})^{-1}P_{12} \]
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

$S_i \geq 0$
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

$S_i \geq 0$

$S_i$ is irreducible
Inherited Properties

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$S_i$ is irreducible

$\rho(S_i) = \rho(P) = \rho$
Inherited Properties

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$\rho(S_i) = \rho(P) = \rho$

For $P$ stochastic

$S_i$ is stochastic

$S_i$ represents a censored Markov chain
Inherited Properties

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For $P$ stochastic

$S_i$ is stochastic

$S_i$ represents a censored Markov chain

Censored Perron vectors

$s_i^T = \text{Left-hand Perron vector for } S_i$

$s_i^T S_i = \rho \ s_i^T$
Aggregation

Objective

Use $s_1^T$ $s_2^T$ ... to build $\pi^T$
Aggregation

Objective

Use $s_1^T s_2^T \cdots$ to build $\pi^T$

Aggregation Matrix

$$A = \begin{bmatrix} s_1^T P_{11} e & s_1^T P_{12} e \\ s_2^T P_{21} e & s_2^T P_{22} e \end{bmatrix}_{2 \times 2}$$
Aggregation

Objective

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Aggregation

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Aggregation Matrix

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A = \begin{bmatrix}
  s_1^T P_{11} e & s_1^T P_{12} e \\
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\end{bmatrix}_{2 \times 2}
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Inherited Properties

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\( A \) is irreducible
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$A$ is irreducible

$\rho(A) = \rho = \rho(P) = \rho(S_i)$
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Inherited Properties

$A \geq 0$

$A$ is irreducible

$\rho(A) = \rho = \rho(P) = \rho(S_i)$

$P$ stochastic $\implies$ $A$ stochastic
Disaggregation

The A / D Theorem

If

\[ s_i^T = \text{Perron vectors for } S_i = P_{ii} + P_{i*}(\rho I - P_{**})^{-1}P_{*i} \]

\[ \alpha^T = (\alpha_1, \alpha_2) = \text{Perron vector for } A = \begin{bmatrix} s_1^T P_{11} e & s_1^T P_{12} e \\ s_2^T P_{21} e & s_2^T P_{22} e \end{bmatrix}_{2 \times 2} \]

then

\[ \pi^T = (\alpha_1 s_1^T \mid \alpha_2 s_2^T) = \text{Perron vector for } P_{n \times n} \]
Disaggregation

The A / D Theorem

If

\[ s_i^T = \text{Perron vectors for } S_i = P_{ii} + P_{i*}(\rho I - P_{**})^{-1}P_{*i} \]

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then

\[ \pi^T = \begin{pmatrix} \alpha_1 s_1^T \\ \alpha_2 s_2^T \end{pmatrix} = \text{Perron vector for } P_{n \times n} \]

Corollary

\[ s_1^T = (\pi_1, \ldots, \pi_g) / \sum_{i=1}^g \pi_i \]

\[ s_2^T = (\pi_{g+1}, \ldots, \pi_n) / \sum_{i=g+1}^n \pi_i \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix} \quad \text{(known)} \]

\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \quad \text{(known)} \]
Updating By Aggregation

Prior Data

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\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \quad \text{(known)} \]

Updated Data

\[ \mathbf{P}_{n \times n} = \text{New Google Matrix} \quad \text{(known)} \]
\[ \pi^T = (\pi_1, \pi_2, \ldots, \pi_n) = \text{New PageRank Vector} \quad \text{(unknown)} \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix} \quad \text{(known)} \]
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Separate Pages Likely To Be Most Affected

\[ G = \{\text{most affected}\} \quad \overline{G} = \{\text{less affected}\} \quad S = G \cup \overline{G} \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix} \quad (\text{known}) \]

\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \quad (\text{known}) \]

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\[ P_{n \times n} = \text{New Google Matrix} \quad (\text{known}) \]

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Separate Pages Likely To Be Most Affected

\[ G = \{\text{most affected}\} \quad \overline{G} = \{\text{less affected}\} \]

\[ S = G \cup \overline{G} \]

New pages (and neighbors) go into \( G \)
Aggregation

Partition

$$P_{n \times n} = \frac{G}{\overline{G}} \left( \begin{array}{cc} G & \overline{G} \\ P_{11} & P_{12} \\ P_{21} & P_{22} \end{array} \right) = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & P_{22} \end{bmatrix}$$

$$\pi^T = (\pi_1, \cdots, \pi_g \mid \pi_{g+1}, \cdots, \pi_n)$$
Aggregation

Partition

\[ P_{n \times n} = \frac{G}{G} \begin{pmatrix} G & \bar{G} \\ P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & P_{22} \end{bmatrix} \]

\[ \pi^T = (\pi_1, \ldots, \pi_g | \pi_{g+1}, \ldots, \pi_n) \]

Perron Complements

\[ p_{11} \cdots p_{gg} \text{ are } 1 \times 1 \implies \text{Perron complements} = 1 \]

\[ \implies \text{Perron vectors} = 1 \]
Aggregation

**Partition**

\[
P_{n \times n} = \frac{G}{\bar{G}} \begin{pmatrix} G & \bar{G} \\ P_{11} & P_{12} \\ \bar{P}_{21} & \bar{P}_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & \bar{P}_{22} \end{bmatrix}
\]

\[
\pi^T = (\pi_1, \ldots \pi_g | \pi_{g+1}, \ldots, \pi_n)
\]

**Perron Complements**

\(p_{11} \cdots p_{gg}\) are \(1 \times 1\) \(\Rightarrow\) Perron complements \(= 1\)

\(\Rightarrow\) Perron vectors \(= 1\)

One significant complement \(S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}\)
Aggregation

Partition

\[ \mathbf{P}_{n \times n} = \frac{G}{\overline{G}} \begin{pmatrix} G & \overline{G} \\ P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & \mathbf{P}_{22} \end{bmatrix} \]

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\[ p_{11}, \ldots, p_{gg} \text{ are } 1 \times 1 \quad \Rightarrow \quad \text{Perron complements} = 1 \]

\[ \Rightarrow \quad \text{Perron vectors} = 1 \]

One significant complement \( \mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12} \)

One significant Perron vector \( \mathbf{s}_2^T \mathbf{S}_2 = \mathbf{s}_2^T \)
Aggregation

**Partition**

\[ P_{n \times n} = \frac{G}{G} \begin{pmatrix} G & \overline{G} \\ P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_{1}^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_{g}^T \\ c_{1} & \cdots & c_{g} & P_{22} \end{bmatrix} \]

\[ \pi^T = (\pi_1, \ldots, \pi_g \mid \pi_{g+1}, \ldots, \pi_n) \]

**Perron Complements**

- \( p_{11} \cdots p_{gg} \) are \( 1 \times 1 \) \( \Rightarrow \) Perron complements = 1
- \( \Rightarrow \) Perron vectors = 1

One significant complement \( S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12} \)

One significant Perron vector \( s_2^T S_2 = s_2^T \)

A/D corollary \( \Rightarrow \) \( s_2^T = (\pi_{g+1}, \ldots, \pi_n) / \sum_{i=g+1}^{n} \pi_i \)
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\((\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)\)

By A/D Corollary

\[ s_{T_2} = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^{n} \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^{n} \phi_i} \equiv \tilde{s}_{T_2} \]
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\[(\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)\]

By A/D Corollary

\[
\begin{align*}
\mathbf{s}_2^T &= \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^n \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^n \phi_i} \equiv \tilde{\mathbf{s}}_2^T \\
\end{align*}
\]

Approximate Aggregation Matrix

\[
\tilde{\mathbf{A}} \equiv \begin{bmatrix}
\mathbf{P}_{11} & \mathbf{P}_{12} \mathbf{e} \\
\tilde{\mathbf{s}}_2^T \mathbf{P}_{21} & \tilde{\mathbf{s}}_2^T \mathbf{P}_{22} \mathbf{e}
\end{bmatrix}_{g+1 \times g+1}
\]
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\[(\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)\]  
(the smaller ones)

By A/D Corollary

\[s_T^2 = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^n \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^n \phi_i} \equiv s_T^2\]

Approximate Aggregation Matrix

\[\tilde{A} \equiv \begin{bmatrix} P_{11} & P_{12}e \\ \tilde{s}_2^T P_{21} & \tilde{s}_2^T P_{22} e \end{bmatrix}_{g+1 \times g+1}\]

\[\tilde{\alpha}^T = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1})\]

By A/D Theorem

\[\tilde{\pi}^T \equiv (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g | \tilde{\alpha}_{g+1} \tilde{s}_2^T) \approx \pi^T\]  
(not bad)
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges

Solution

Perturb A/D output to move off of fixed point
Move in direction of solution
\[ \tilde{\pi}^T = \tilde{\pi}^T P \]

(a smoothing step)
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges

Solution

Perturb A/D output to move off of fixed point
Move in direction of solution

\[ \tilde{\pi}^T = \tilde{\pi}^T P \]  
(a smoothing step)

The Iterative A/D Updating Algorithm

Determine the “G-set” partition \( S = G \cup \overline{G} \)
Approximate A/D step generates \( \tilde{\pi}^T \)
Smooth \( \tilde{\pi}^T = \tilde{\pi}^T P \)
Use \( \tilde{\pi}^T \) as input to another approximate aggregation step

\vdots
Convergence

THEOREM

Always converges to the new PageRank vector $\pi^T$
Convergence

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Converges for all partitions $S = G \cup \overline{G}$
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Rate of convergence governed by $|\lambda_2(S_2)|$

$$S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$
Convergence

**THEOREM**

Always converges to the new PageRank vector $\pi^T$

Converges for all partitions $S = G \cup \overline{G}$

Rate of convergence governed by $|\lambda_2(S_2)|$

$$S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$

**THE GAME**

Find a relatively small $G$ to minimize $|\lambda_2(S_2)|$
# Experiments

## Test Networks From Crawl Of Web

<table>
<thead>
<tr>
<th>Category</th>
<th>Nodes</th>
<th>Links</th>
</tr>
</thead>
<tbody>
<tr>
<td>Censorship</td>
<td>562 nodes</td>
<td>736 links</td>
</tr>
<tr>
<td>Movies</td>
<td>451 nodes</td>
<td>713 links</td>
</tr>
<tr>
<td>MathWorks</td>
<td>517 nodes</td>
<td>13,531 links</td>
</tr>
<tr>
<td>Abortion</td>
<td>1,693 nodes</td>
<td>4,325 links</td>
</tr>
<tr>
<td>Genetics</td>
<td>2,952 nodes</td>
<td>6,485 links</td>
</tr>
<tr>
<td>California</td>
<td>9,664 nodes</td>
<td>16,150 links</td>
</tr>
</tbody>
</table>
Perturbations

The Updates

# Nodes Added = 3
# Nodes Removed = 50
# Links Added = 10
# Links Removed = 20

(Different values have little effect on results)

Stopping Criterion

1-norm of residual < $10^{-10}$
### Movies

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterative Aggregation</th>
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<tbody>
<tr>
<td>Iterations</td>
<td>Time</td>
</tr>
<tr>
<td>17</td>
<td>.40</td>
</tr>
<tr>
<td>10</td>
<td>12</td>
</tr>
<tr>
<td>15</td>
<td>11</td>
</tr>
<tr>
<td>20</td>
<td>11</td>
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<tr>
<td>100</td>
<td>9</td>
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<tr>
<td>200</td>
<td>8</td>
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<tr>
<td>300</td>
<td>7</td>
</tr>
<tr>
<td>400</td>
<td>6</td>
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$nodes = 451 \quad links = 713$
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| Iterative Aggregation | \(|G|\) | Iterations | Time |
|-----------------------|--------|------------|------|
|                       | 5      | 12         | .39  |
|                       | 10     | 12         | .37  |
|                       | 15     | 11         | .36  |
|                       | 20     | 11         | .35  |
|                       | 25     | 11         | .31  |
|                       | 50     | 9          | .31  |
|                       | 100    | 9          | .33  |
|                       | 200    | 8          | .35  |
|                       | 300    | 7          | .39  |
|                       | 400    | 6          | .47  |

\(nodes = 451\) \(links = 713\)
## Censorship

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<td></td>
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| Iterative Aggregation | $|G|$ | Iterations | Time |
|-----------------------|-----|------------|------|
|                       | 5   | 38         | 1.68 |
|                       | 10  | 38         | 1.66 |
|                       | 15  | 38         | 1.56 |
|                       | 20  | 20         | 1.06 |
|                       | 25  | 20         | 1.05 |
|                       | 50  | 10         | 0.69 |
|                       | 100 | 8          | 0.55 |

\[ \text{nodes} = 562 \quad \text{links} = 736 \]
## Censorship

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$nodes = 562 \quad links = 736$
### MathWorks

#### Power Method

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</table>

#### Iterative Aggregation

| $|G|$ | Iterations | Time  |
|-----|------------|-------|
| 5   | 53         | 1.18  |
| 10  | 52         | 1.29  |
| 15  | 52         | 1.23  |
| 20  | 42         | 1.05  |
| 25  | 20         | 1.13  |
| 300 | 11         | .83   |
| 400 | 10         | 1.01  |

*nodes* = 517  *links* = 13,531
## Power Method

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## Iterative Aggregation

| $|G|$ | Iterations | Time |
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| 5  | 53         | 1.18 |
| 10 | 52         | 1.29 |
| 15 | 52         | 1.23 |
| 20 | 42         | 1.05 |
| 25 | 20         | 1.13 |
| 50 | 18         | .70  |
| 100| 16         | .70  |
| 200| 13         | .70  |
| 300| 11         | .83  |
| 400| 10         | 1.01 |

*nodes = 517  links = 13,531*
## Abortion

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterative Aggregation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Iterations</strong></td>
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<tr>
<td>106</td>
<td>5</td>
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<td>500</td>
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<tr>
<td></td>
<td>750</td>
</tr>
<tr>
<td></td>
<td>1000</td>
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</tbody>
</table>

\( \text{nodes} = 1,693 \quad \text{links} = 4,325 \)
<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterative Aggregation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Iterations</strong></td>
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<tr>
<td>106</td>
<td>37.08</td>
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</tbody>
</table>

\[ \text{nodes} = 1,693 \quad \text{links} = 4,325 \]
## Genetics

<table>
<thead>
<tr>
<th>Power Method</th>
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<tbody>
<tr>
<td><strong>Iterations</strong></td>
<td><strong>Time</strong></td>
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<tr>
<td>92</td>
<td>91.78</td>
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<td>13</td>
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<td>1000</td>
<td>5</td>
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<tr>
<td>1500</td>
<td>5</td>
</tr>
</tbody>
</table>

*nodes = 2,952  links = 6,485*
## Genetics

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<tr>
<th>Power Method</th>
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<td>5</td>
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</table>

nodes = 2,952  
links = 6,485
<table>
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<th>Power Method</th>
<th>Iterative Aggregation</th>
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</thead>
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<tr>
<td>Iterations</td>
<td>Time</td>
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<tr>
<td>176</td>
<td>5.85</td>
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</table>

$nodes = 9,664 \quad links = 16,150$
## California

### Power Method

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>176</td>
<td>5.85</td>
</tr>
</tbody>
</table>

### Iterative Aggregation

| $|G|$  | Iterations | Time |
|------|-------------|------|
| 500  | 19          | 1.12 |
| 1000 | 15          | .92  |
| 1250 | 20          | 1.04 |
| 1500 | 14          | .90  |
| 2000 | 13          | 1.17 |
| 5000 | 6           | 1.25 |

\[ \text{nodes} = 9,664 \quad \text{links} = 16,150 \]
“L” Curves

- Censorship
- Movies
- Mathworks
- Abortion
- Genetic
- CA
Comparisons

Race

→ Power Method

→ Power Method + Quadratic Extrapolation

→ Iterative Aggregation

→ Iterative Aggregation + Quadratic Extrapolation
Comparisons

Race
→ Power Method
→ Power Method + Quadratic Extrapolation
→ Iterative Aggregation
→ Iterative Aggregation + Quadratic Extrapolation

NC State Internal Crawl
→ 10,000 nodes + 101,118 links
  → 50 nodes added
  → 30 nodes removed
    → 300 links added
    → 200 links removed
The graph shows the convergence of the power method, plotted as \( \log_{10} \text{residual} \) against iteration number. The data points form a clear downward trend, indicating rapid convergence of the algorithm.

**Axes:**
- **Y-axis:** \( \log_{10} \text{residual} \)
- **X-axis:** Iteration

**Legend:**
- Power method represented by blue stars.
Iterations

**Power method**

**Power method with quad. extrap.**
Iterations

- Power method
- Power method with quadratic extrapolation
- I A D

Graph shows the logarithm of the residual against iterations.
Iterations

- Power method
- Power method with quad. extrap.
- IAD
- IAD with quad. extrap.
## Timings

|          | Iterations | Time | $|G|$ |
|----------|------------|------|-----|
| Power    | 162        | 9.69 |     |
| Power+Quad |           |      |     |
| IAD      |            |      |     |
| IAD+Quad |            |      |     |

$nodes = 10,000 \quad links = 101,118$
## Timings

| Iterations | Time  | $|G|$ |
|------------|-------|-----|
| **Power**  | 162   | 9.69|
| **Power+Quad** | 81   | 5.93|
| **IAD**    |       |     |
| **IAD+Quad** |      |     |

$nodes = 10,000 \quad links = 101,118$
## Timings

|                      | Iterations | Time  | $|G|$ |
|----------------------|------------|-------|-----|
| Power                | 162        | 9.69  |     |
| Power+Quad           | 81         | 5.93  |     |
| IAD                  | 21         | 2.22  | 2000|
| IAD+Quad             |            |       |     |

$\text{nodes} = 10,000 \quad \text{links} = 101,118$
## Timings

|          | Iterations | Time  | $|G|$, $n = 10,000$, $m = 101,118$ |
|----------|------------|-------|---------------------------------|
| Power    | 162        | 9.69  |                                |
| Power+Quad | 81        | 5.93  |                                |
| IAD      | 21         | 2.22  | 2000                            |
| IAD+Quad | 16         | 1.85  | 2000                            |

*nodes* = 10,000  *links* = 101,118
Conclusion

Iterative A/D with appropriate partitioning and smoothing shows promise for updating Markov chains with power law distributions.