Updating the Stationary Distribution Vector for an Irreducible Markov Chain

Prakash Chanchana

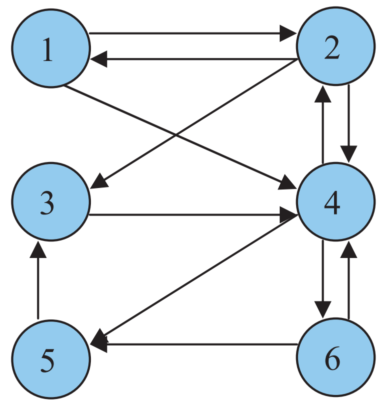
Carl D. Meyer

Amy N. Langville

Outline

- PageRank Problem
- Updating Problem
- Exact Aggregation
- Approximate Updating Algorithm
- Updating Algorithm with the Iterative Aggregation
- Determining the Partition
- Future Work

PageRank Problem

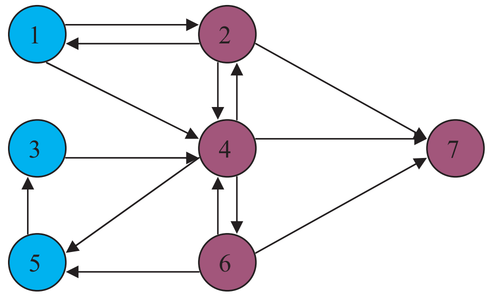


$$Q = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 0 \end{pmatrix}_{6\times 6}$$

Solving for the dominant eigenvector of a matrix Q by using the **Power Method**.

$$\phi^T Q = \phi^T$$

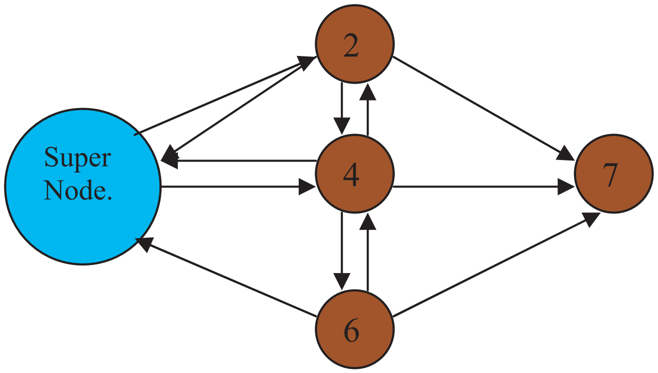
Updating PageRank Problem



$$P = \begin{pmatrix} 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1/4 & 0 & 0 & 1/4 & 1/4 & 1/4 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{7\times 7}$$

- Given ϕ^T and Q for the original system, find π^T of P for updated system.
- One way to solve this problem is to apply the Power Method (not good for a **Huge** matrix)

Motivation



- Working with smaller system
- Paying more attention to the updated states

Exact Aggregation

1. Partition states into k disjoint groups as $S = G_1 \cup G_2 \cup \ldots \cup G_k$ where each P_{ii} is a square block.

$$P_{n\times n} = \begin{pmatrix} P_{11} & P_{12} & \ldots & P_{1k} \\ P_{21} & P_{22} & \ldots & P_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ P_{k1} & P_{n2} & \ldots & P_{kk} \end{pmatrix}$$

2. Find the stochastic complement S_i for each block P_{ii} where

$$S_i = P_{ii} + P_{i*}(I - P_i^*)^{-1} P_{*i}$$

-P_{i*} and P_{*i} are, respectively, the i^{th}-row and the i^{th}-column of block with P_{ii} removed.

-P_i^* is principle submatrix of P obtained by deleting the i^{th}-row and the i^{th}-column of blocks.

Exact Aggregation

Example:

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}, then$$

$$S_1 = P_{11} + P_{12}(I - P_{22})^{-1} P_{21}$$

$$S_2 = P_{22} + P_{21}(I - P_{11})^{-1} P_{12}$$

3. Form the smaller $k \times k$ matrix called the *Aggregated Transition Matrix.*

$$A_{k\times k} = \begin{pmatrix} s_1^T P_{11} e & \dots & s_1^T P_{1k} e \\ \vdots & \ddots & \vdots \\ s_k^T P_{k1} e & \dots & s_k^T P_{kk} e \end{pmatrix}$$

where s_i^T is the stationary probability vector of S_i.

Exact Aggregation

4. Find the stationary distribution of A: $\alpha^T = (\alpha_1, \alpha_2, \ldots, \alpha_k)$

5. By the Exact Aggregation Theorem: $\pi^T = (\alpha_1 s_1^T, \alpha_2 s_2^T, \ldots, \alpha_k s_k^T)$

Advantage : Get the exact value of π^T

Disadvantage : Very expensive for computing inverse $S_i = P_{ii} + P_{i*}(I - P_i^*)^{-1} P_{*i}$

Special Partition

$$P = \begin{array}{c c} & \begin{array}{cc} G & \bar{G} \end{array} \\ \begin{array}{c} G \\ \bar{G} \end{array} & \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \end{array}$$

Partition P into $g+1$ levels where first g diagonal blocks are 1×1

$$P = \left(\begin{array}{c|c|c|c} p_{11} & \dots & p_{1g} & P_{1*} \\ \hline \vdots & \ddots & \vdots & \vdots \\ \hline p_{g1} & \dots & p_{gg} & P_{g*} \\ \hline P_{*1} & \dots & P_{*g} & P_{22} \end{array}\right)$$

Advantage of this partition:

- Since $p_{11} \dots p_{gg}$ are 1×1 block $\Rightarrow$ Stochastic complement = 1 and $s_i = 1$

Special Partition

- There is only one stochastic complement

$$S = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$

- By Exact Aggregation Theorem

$$\tilde{s}^T = (\phi_{g+1}, \ldots, \phi_n)/\Sigma_{i=g+1}^{n}\phi_i$$

So, the Exact Aggregated Transition Matrix associated with the new partition is

$$\tilde{A} = \left(\begin{array}{ccc|c} p_{11} & \ldots & p_{1g} & P_{1*}e \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \ldots & p_{gg} & P_{g*}e \\ \hline \tilde{s}^T P_{*1} & \ldots & \tilde{s}^T P_{*g} & \tilde{s}^T P_{22}e \end{array}\right)_{(g+1)\times(g+1)}$$

Special Partition

Since we use the old stationary vector ϕ^T to find the updated stationary vector π^T

$$\Rightarrow \pi^T \approx \phi^T$$

$$A \approx \tilde{A} = \begin{pmatrix} P_{11} & P_{12}e \\ \tilde{s}^T P_{21} & \tilde{s}^T P_{22}e \end{pmatrix}$$

$$\alpha^T \approx \tilde{\alpha}^T = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1})$$

$$\pi^T \approx \tilde{\pi}^T = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1}\tilde{s}^T)$$

(not bad approximation)

Approximate Aggregation Algorithm

1. Partition states of updated as $S = G \cup \bar{G}$ and reorder $P = \begin{array}{c c} & \begin{array}{cc} G & \bar{G} \end{array} \\ \begin{array}{c} G \\ \bar{G} \end{array} & \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \end{array}$
2. Set $\bar{\phi}^T = (\phi_{g+1}, \phi_{g+2} \ldots, \phi_n)$
3. Set $\tilde{s}^T = \bar{\phi}^T / \bar{\phi}^T e$ where e is a column of ones
4. Set $\tilde{A} = \begin{pmatrix} P_{11} & P_{12}e \\ \tilde{s}^T P_{21} & 1 - \tilde{s}^T P_{21} \end{pmatrix}_{(g+1)\times(g+1)}$
5. Find $\tilde{\alpha}^T = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1})$
6. Set $\tilde{\pi}^T = (\tilde{\alpha}_1, \tilde{\alpha}_2, \ldots, \tilde{\alpha}_g | \bar{\phi}^T)$

Iterative Aggregation Algorithm

Initialization

- Partition the states of the updated chain as $S = G \cup \bar{G}$
- $\bar{\phi}^T \leftarrow$ the component from ϕ^T corresponding to the states in $\bar{G}$
- $s^T \leftarrow \bar{\phi}^T / (\bar{\phi}^T e)$

Iterate until convergence

1. $\tilde{A} \leftarrow \begin{pmatrix} P_{11} & P_{12}e \\ \tilde{s}^T P_{21} & 1 - \tilde{s}^T P_{21} \end{pmatrix}$
2. $\alpha^T \leftarrow (\alpha_1, \alpha_2, \ldots, \alpha_g, \alpha_{g+1})$
3. $\chi^T \leftarrow (\alpha_1, \alpha_2, \ldots, \alpha_g | \alpha_{g+1} s^T)$
4. $\psi^T \leftarrow \chi^T$**P** (Move the iterates off the fixed point$\longrightarrow$ the convergence.)
5. If $\|\psi^T - \chi^T\| < \tau$ for a given tolerance τ , then quit
   else $s^T \leftarrow \psi^T / \psi^T e$ go back to step 1.

How to determine the partition for the updating problem ?

Rules of the partition

- Put the updated states into G.
- Put the nearest states to the update into G.
- Put the states which have a large stationary probability into G.(slow converging states shown by Kamvar et al.)

Experiments

Topic	Number of Nodes	Number of Links	Total number of nodes	Total number of links
Genetics	2,952	6,485	3,200	6,900
California	9,664	16,150	10,300	17,000

- Adding and deleting some nodes
- Adding and deleting some links
- Stopping criterion $\|\psi^T - \chi^T\|_1 < 10^{-10}$

Genetics

$g = \|G\|$	$Iterations$	$Time\ (sec)$
10	163	2.16
50	19	.483
100	19	.456
250	17	.276
500	9	.313
1000	8	.319
$Power Method$	165	1.45

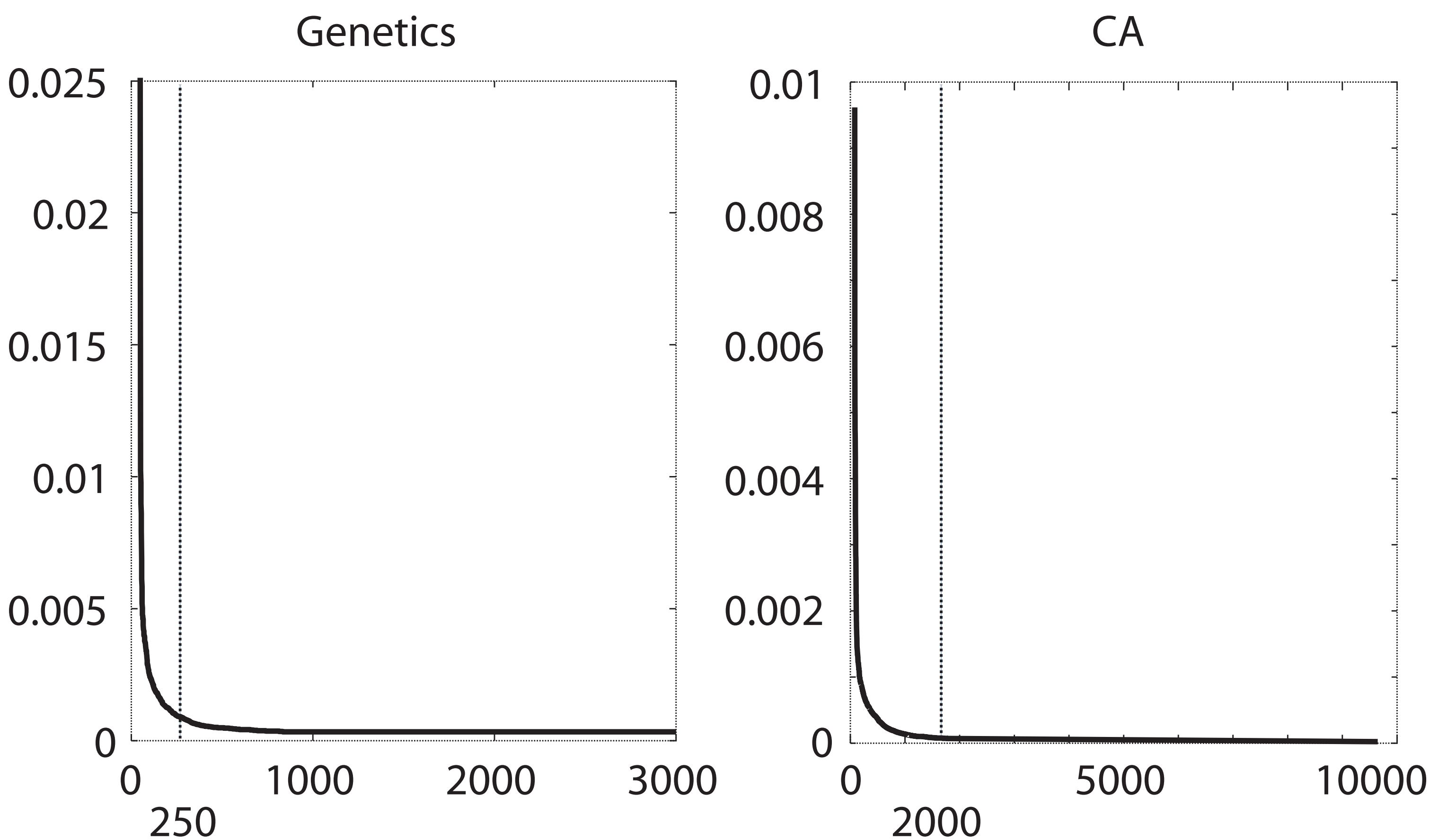
Iterative aggregation algorithm is about 5 times faster than Power method

California

$g = \|G\|$	$Iterations$	$Time\ (sec)$
10	170	7.75
50	75	3.56
100	57	3.75
250	51	2.59
500	34	2.01
1000	19	1.03
2000	10	.997
3000	7	1.17
4000	7	1.22
5000	7	1.56
$PowerMethod$	176	5.87

Iterative aggregation algorithm is about 6 times faster than Power method

Location of G-Opt



- Size of G near optimal stay around the right of the bend.

Conclusion And Future Work

Conclusion

Appropriate partitions can greatly speed up Stationary vector computation.

Future work

- Combine the Iterative Aggregation with other methods to increase the speed of entire process.
- Find a concrete way to partition $S = G \cup \bar{G}$.
- Investigate the relationship between $|\lambda_2|$ of the **S** and $|G|$.
- Work with real life data.