



Updating Markov Chains

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Intro

Assumptions

Very large irreducible chain

— $m = O(10^9)$

number of states

— $\mathbf{Q}_{m \times m}$

old transition matrix

— $\phi^T = (\phi_1, \phi_2, \dots, \phi_m)$

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Updates

Change some transition probabilities

Add or delete some states

- $\mathbf{P}_{n \times n}$ new transition matrix (irreducible)
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Aim

Use ϕ^T to compute π^T



Exact (Theoretical) Updating

Perturbation Formula

$$\mathbf{P} = \mathbf{Q} - \mathbf{E} \implies \boldsymbol{\pi}^T = \boldsymbol{\phi}^T - \boldsymbol{\epsilon}^T$$



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$$\mathbf{Z} = \begin{cases} \text{Fundamental Matrix} \\ (\mathbf{I} - \mathbf{Q})^\# \quad (\text{group inverse}) \end{cases}$$



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$$\mathbf{p}_i^T = \mathbf{q}_i^T - \boldsymbol{\delta}_i^T \implies \boldsymbol{\pi}^T = \boldsymbol{\phi}^T - \boldsymbol{\epsilon}^T$$



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Not Practical For Large Problems



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$$\mathbf{x}_{j+1}^T = \mathbf{x}_j^T \mathbf{P} \quad \text{with} \quad \mathbf{x}_0^T = \phi^T$$

(assume aperiodic)



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- ▷ Want 12 digits of accuracy



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A Little Better, But Not Great



Censoring

Partition (not necessarily NCD)

$$\mathbf{P}_{n \times n} = \begin{matrix} & G_1 & G_2 & \cdots & G_k \\ G_1 & \mathbf{P}_{11} & \mathbf{P}_{12} & \cdots & \mathbf{P}_{1k} \\ G_2 & \mathbf{P}_{21} & \mathbf{P}_{22} & \cdots & \mathbf{P}_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ G_k & \mathbf{P}_{k1} & \mathbf{P}_{k2} & \cdots & \mathbf{P}_{kk} \end{matrix}$$



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Records state of process only when chain visits states in G_i .

Visits to states outside of G_i are ignored.



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Censored Transition Matrices

$$\mathbf{C}_i = \mathbf{P}_{ii} + \mathbf{P}_{i\star}(\mathbf{I} - \mathbf{P}_i^{\star})^{-1}\mathbf{P}_{\star i}$$

Stochastic Complements



Aggregation

Censored Distributions

$$\mathbf{s}_i^T \mathbf{C}_i = \mathbf{s}_i^T$$



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$$\mathcal{A}_{k \times k} = \begin{bmatrix} \mathbf{s}_1^T \mathbf{P}_{11} \mathbf{e} & \cdots & \mathbf{s}_1^T \mathbf{P}_{1k} \mathbf{e} \\ \vdots & \ddots & \vdots \\ \mathbf{s}_k^T \mathbf{P}_{k1} \mathbf{e} & \cdots & \mathbf{s}_k^T \mathbf{P}_{kk} \mathbf{e} \end{bmatrix}$$

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Aggregation Theorem

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Intuition

Update relatively small number of states in large sparse chain

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Induced Matrix Partition

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Specialized Aggregation

Censored Transition Matrices

$$\mathbf{C}_1 = \cdots = \mathbf{C}_g = [1]_{1 \times 1} \quad \mathbf{C}_{g+1} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$



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$$\mathbf{A} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}_{g+1}^T \mathbf{P}_{21} & 1 - \mathbf{s}_{g+1}^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)}$$

Aggregated Distribution

$$\alpha^T = (\alpha_1, \dots, \alpha_g, \alpha_{g+1})$$

Aggregation Theorem

$$\pi^T = (\pi_1, \dots, \pi_g \mid \bar{\pi}^T) = (\alpha_1, \dots, \alpha_g, \mid \alpha_{g+1} \mathbf{s}_{g+1}^T)$$

Old Distribution (reordered)

$$\phi^T = (\phi_1, \phi_2, \dots \mid \bar{\phi}^T)$$

The Assumption

$$\bar{\phi}^T \approx \bar{\pi}^T \implies \bar{\phi}^T \approx \alpha_{g+1} \mathbf{s}_{g+1}^T \implies \mathbf{s}_{g+1}^T \approx \frac{\bar{\phi}^T}{\bar{\phi}^T \mathbf{e}}$$



Specialized Aggregation

Censored Transition Matrices

$$\mathbf{C}_1 = \dots = \mathbf{C}_g = [1]_{1 \times 1} \quad \mathbf{C}_{g+1} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$$

Censored Distributions

$$\mathbf{s}_1^T = \dots = \mathbf{s}_g^T = \mathbf{1} \quad \mathbf{s}_{g+1}^T = \mathbf{s}_{g+1}^T \mathbf{C}_{g+1}$$

Aggregation Matrix

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Aggregation Theorem

$$\boldsymbol{\pi}^T = (\pi_1, \dots, \pi_g \mid \bar{\boldsymbol{\pi}}^T) = (\alpha_1, \dots, \alpha_g \mid \alpha_{g+1} \mathbf{s}_{g+1}^T)$$

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Old Distribution (reordered)

$$\boldsymbol{\phi}^T = (\phi_1, \phi_2, \dots | \bar{\boldsymbol{\phi}}^T)$$

Updated Distribution

The Assumption

$$\bar{\boldsymbol{\phi}}^T \approx \boldsymbol{\pi}^T \implies \bar{\boldsymbol{\phi}}^T \approx \alpha_{g+1} \mathbf{s}_{g+1}^T \implies \mathbf{s}_{g+1}^T \approx \frac{\bar{\boldsymbol{\phi}}^T}{\boldsymbol{\phi}^T \mathbf{e}}$$



Summary

Reorder & Partition Updated State Space

$$\mathcal{S} = G \cup \overline{G}$$

$$G = \{\text{New + Most affected}\} \quad \overline{G} = \{\text{Less affected}\}$$



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Use Less Affected Components From Old Distribution

$\bar{\phi}^T \leftarrow$ components from ϕ^T corresponding to states in \bar{G}



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$$\mathbf{s}^T \leftarrow \bar{\phi}^T / (\bar{\phi}^T \mathbf{e})$$



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Iterate ?



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Iterate ?

No! — At A Fixed Point



Iterative Aggregation

Move Off Of Fixed Point With Power Step

$$\mathbf{s}^T \leftarrow \overline{\phi}^T / (\overline{\phi}^T \mathbf{e})$$

$$\mathcal{A} \leftarrow \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12}\mathbf{e} \\ \mathbf{s}^T \mathbf{P}_{21} & 1 - \mathbf{s}^T \mathbf{P}_{21}\mathbf{e} \end{bmatrix}_{(g+1) \times (g+1)}$$

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$$\boldsymbol{\pi}^T \leftarrow (\alpha_1, \dots, \alpha_g, | \alpha_{g+1} \mathbf{s}^T)$$

$$\boldsymbol{\psi}^T \leftarrow \boldsymbol{\pi}^T \mathbf{P}$$

(Also makes progress toward convergence when aperiodic)

If $\|\boldsymbol{\psi}^T - \boldsymbol{\chi}^T\| < \tau$ then quit — else

$$\mathbf{s}^T \leftarrow \overline{\boldsymbol{\psi}^T} / (\overline{\boldsymbol{\psi}^T} \mathbf{e})$$



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Theorem

If $\mathbf{C} = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12}$ is aperiodic, then convergent for all partitions $\mathcal{S} = G \cup \overline{G}$



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— $|\lambda_2(\mathbf{C})|$ determines rate of convergence



Google's PageRank

Random Walk On WWW Link Structure

$$\mathbf{H}_{ij} = \begin{cases} 1/(\text{total \# outlinks from page } \mathcal{P}_i) & \text{if } \mathcal{P}_i \rightarrow \mathcal{P}_j, \\ 0 & \text{otherwise} \end{cases}$$



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Google Matrix

$$\mathbf{P} = \alpha(\mathbf{H} + \mathbf{E}) + (1 - \alpha)\mathbf{F}$$

- $(\mathbf{H} + \mathbf{E})$ & \mathbf{F} are stochastic $\text{rank}(\mathbf{E}) = \text{rank}(\mathbf{F}) = 1$
- $0 < \alpha < 1$



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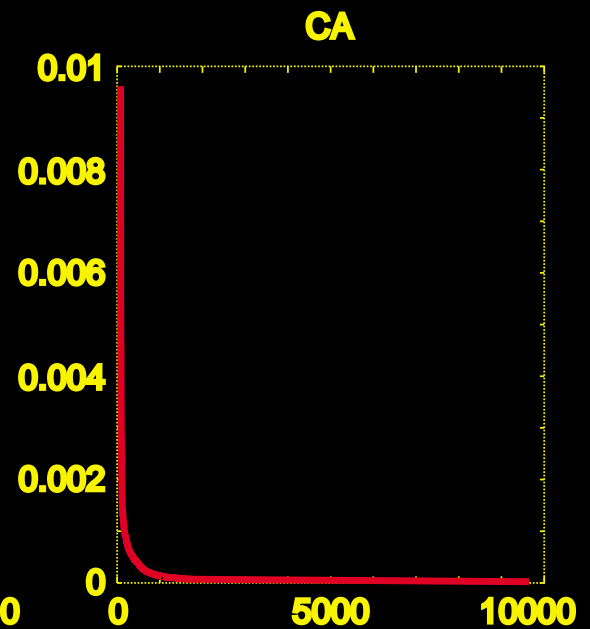
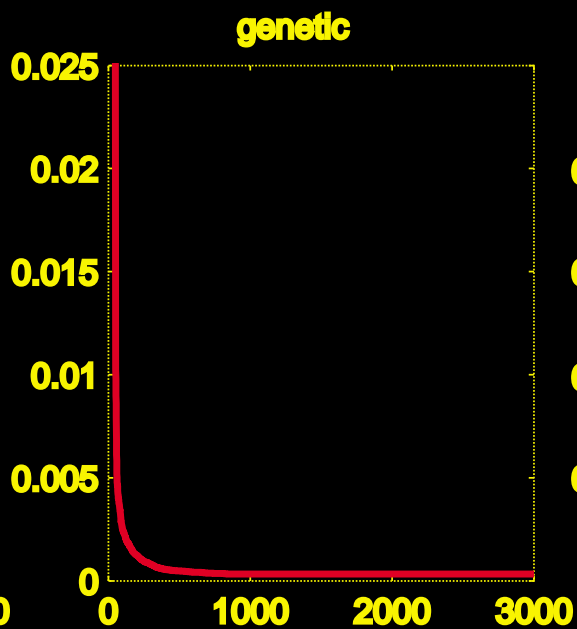
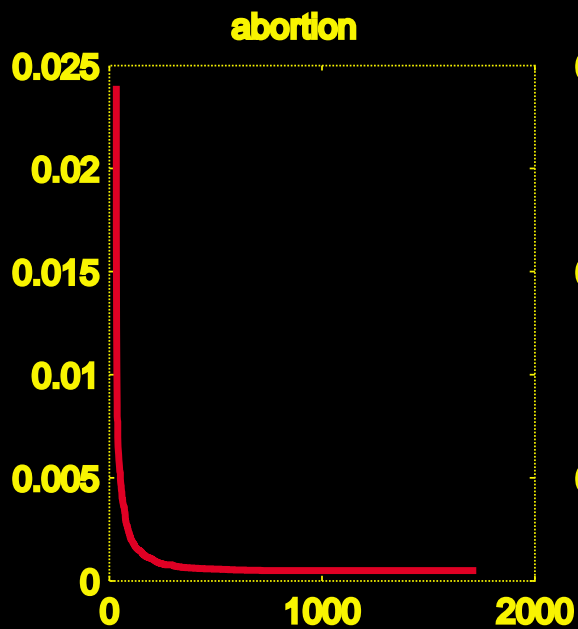
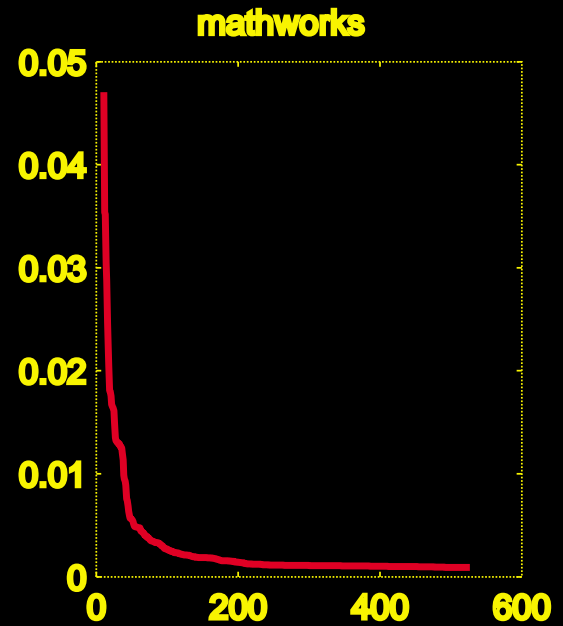
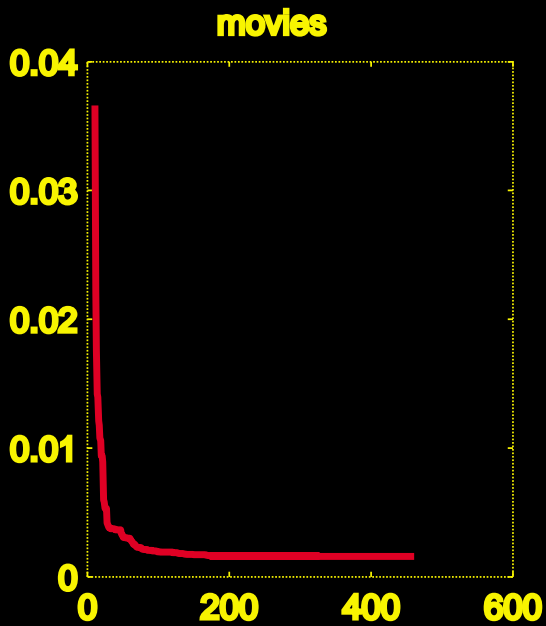
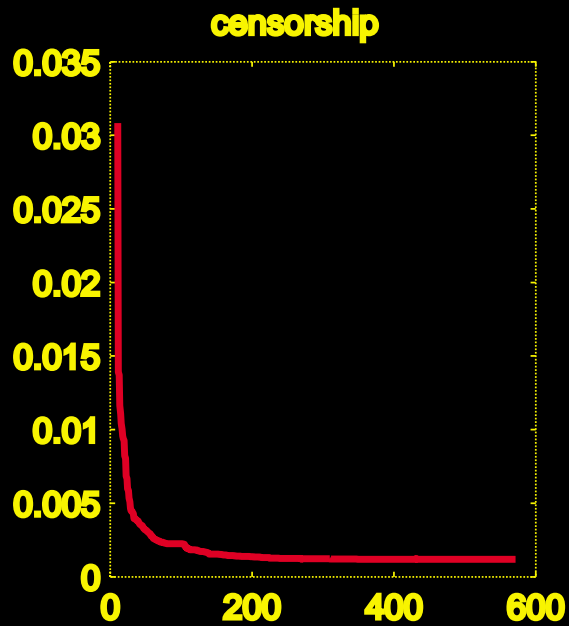
Power Law Distribution

If ordered by magnitude $\pi(1) \geq \pi(2) \geq \dots \geq \pi(n)$, then

- $\pi(i) \approx \alpha i^{-k}$ for $k \approx 2.109$ [Donato, Laura, Leonardi, 2002]
[Pandurangan, Raghavan, & Upfal, 2004]
- Relatively few large states (i.e., important sites)



“L” Curves





Experiments

The Updates

Nodes Added = 3

Nodes Removed = 50

Links Added = 10

(Different values have little effect on results)

Links Removed = 20

Stopping Criterion

1-norm of residual $< 10^{-10}$



Movies

Power Method

<u>Iterations</u>	<u>Time</u>
17	.40

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	12	.39
10	12	.37
15	11	.36
20	11	.35
25	11	.31
50	9	.31
100	9	.33
200	8	.35
300	7	.39
400	6	.47

nodes = 451 links = 713



Censorship

Power Method

<u>Iterations</u>	<u>Time</u>
38	1.40

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	38	1.68
10	38	1.66
15	38	1.56
20	20	1.06
25	20	1.05
50	10	.69
100	8	.55
200	6	.53
300	6	.65
400	5	.70

nodes = 562 links = 736



MathWorks

Power Method

<u>Iterations</u>	<u>Time</u>
54	1.25

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	53	1.18
10	52	1.29
15	52	1.23
20	42	1.05
25	20	1.13
50	18	.70
100	16	.70
200	13	.70
300	11	.83
400	10	1.01

nodes = 517 links = 13,531



Abortion

Power Method

<u>Iterations</u>	<u>Time</u>
106	37.08

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	109	38.56
10	105	36.02
15	107	38.05
20	107	38.45
25	97	34.81
50	53	18.80
100	13	5.18
250	12	5.62
500	6	5.21
750	5	10.22
1000	5	14.61

nodes = 1,693 links = 4,325



Genetics

Power Method

<u>Iterations</u>	<u>Time</u>
92	91.78

Iterative Aggregation

<u>G</u>	<u>Iterations</u>	<u>Time</u>
5	91	88.22
10	92	92.12
20	71	72.53
50	25	25.42
100	19	20.72
250	13	14.97
500	7	11.14
1000	5	17.76
1500	5	31.84

nodes = 2,952 links = 6,485



California

Power Method

Iterations	Time
176	5.85

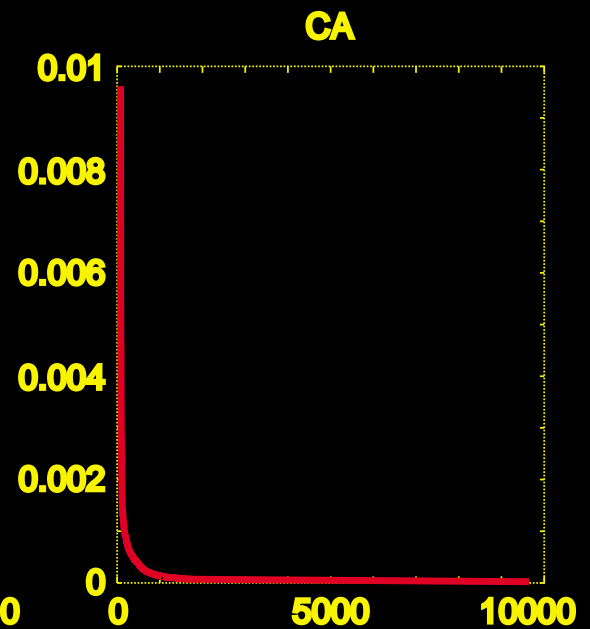
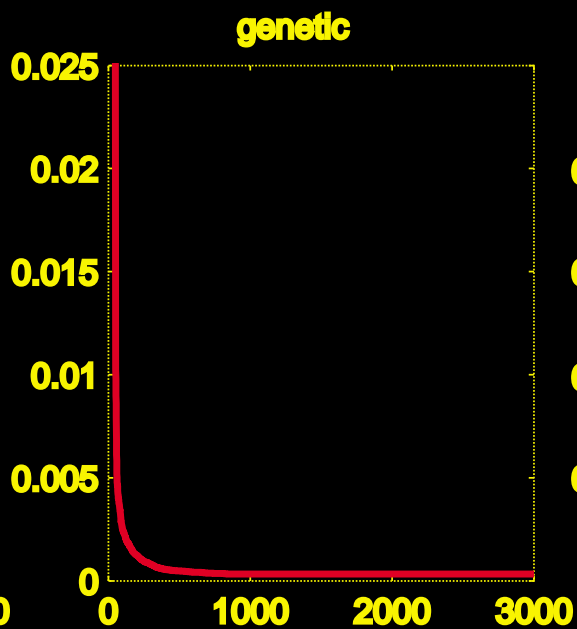
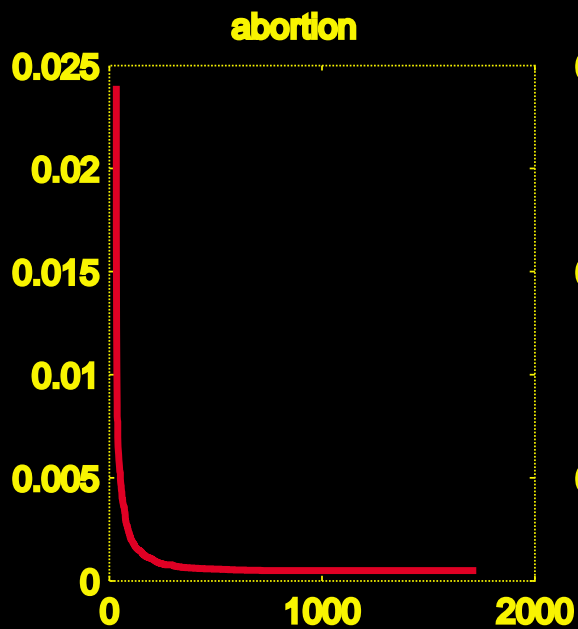
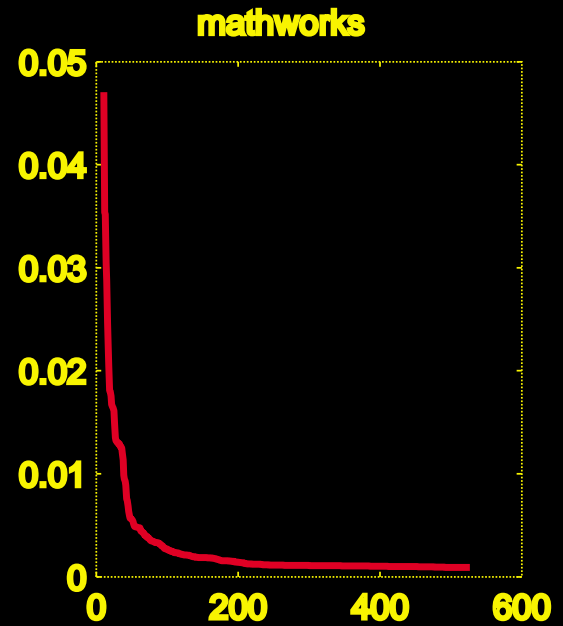
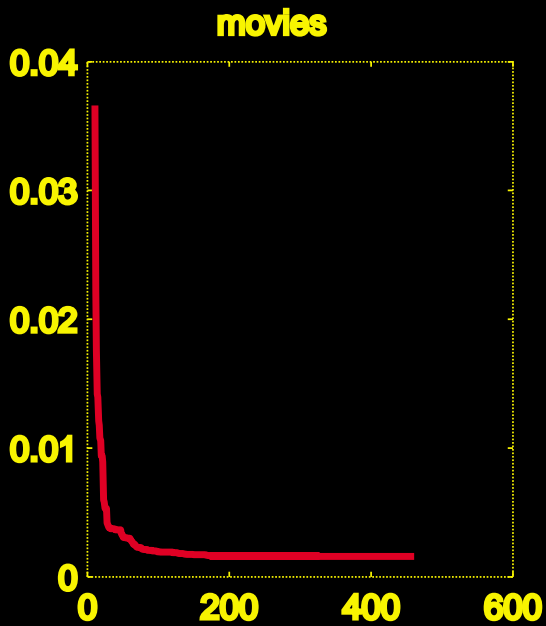
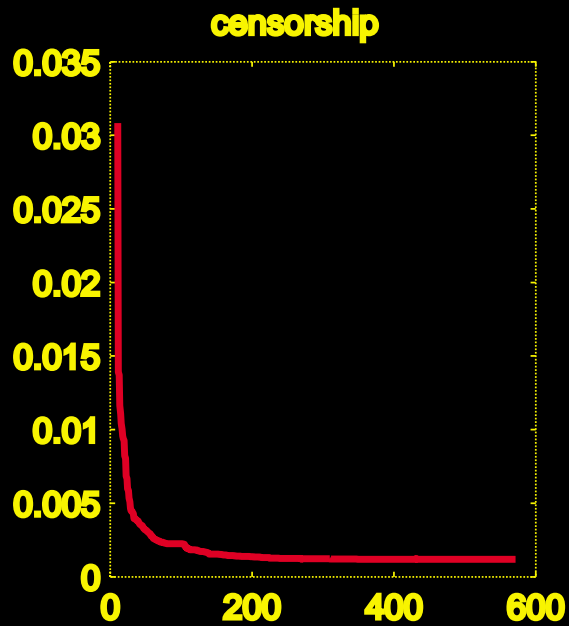
Iterative Aggregation

$ G $	Iterations	Time
500	19	1.12
1000	15	.92
1250	20	1.04
1500	14	.90
2000	13	1.17
5000	6	1.25

nodes = 9,664 links = 16,150

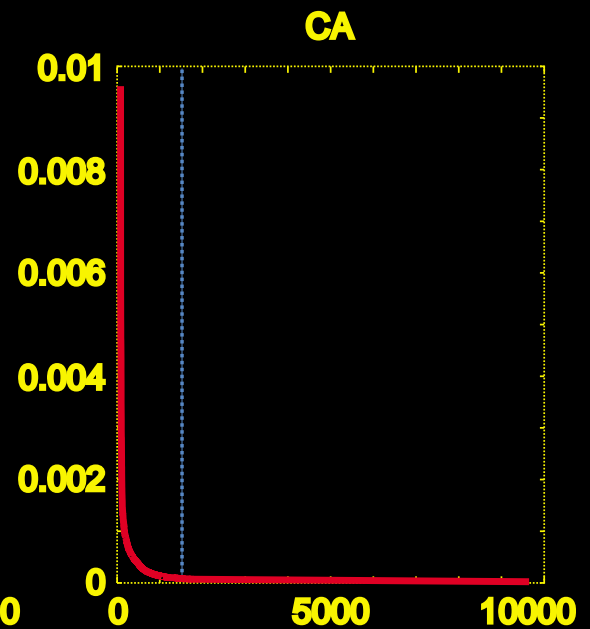
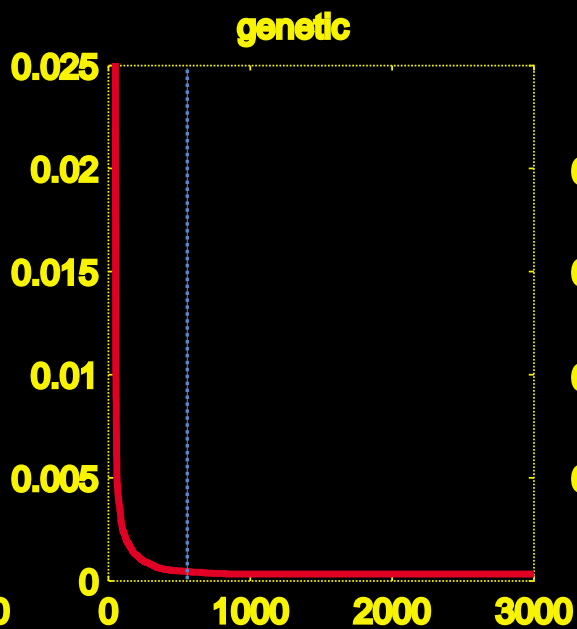
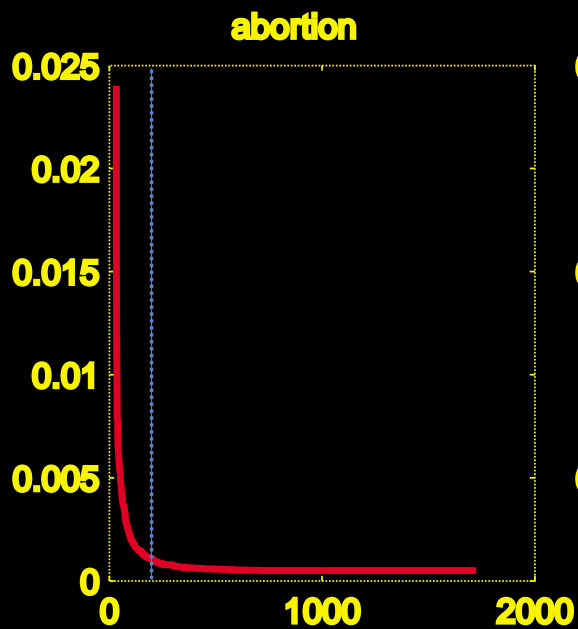
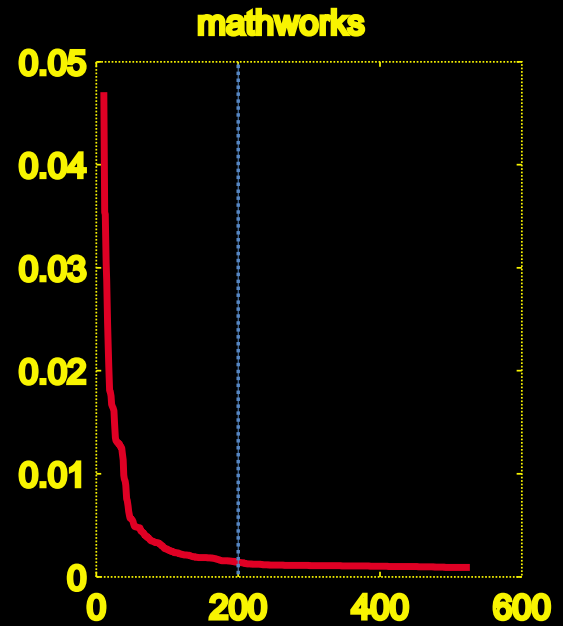
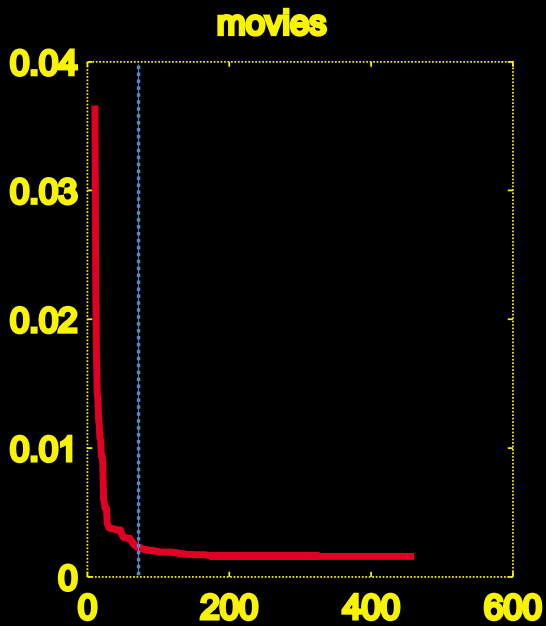
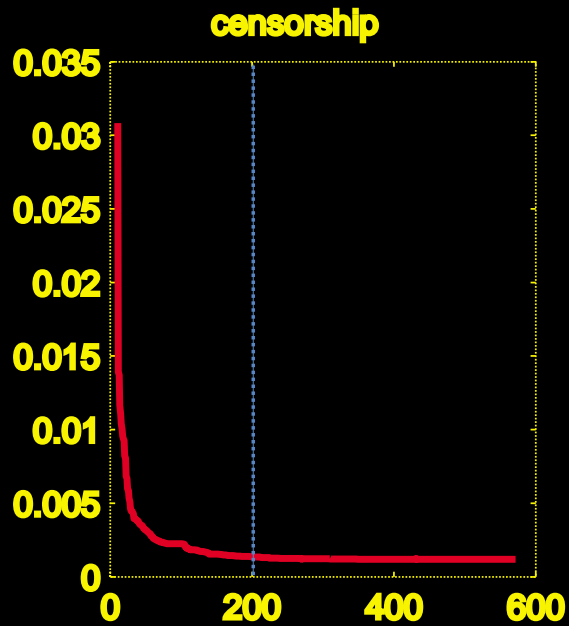


“L” Curves





“L” Curves





Timings

	Iterations	Time (sec)	 G
Power	162	9.69	
Power+Quad	81	5.93	
IAD	21	2.22	2000
IAD+Quad	16	1.85	2000

nodes = 10,000 links = 101,118



Conclusion

- ✦ **Iterative aggregation shows
promise for updating
Markov chains**
- ✦ **Especially for those having
power law distributions**



Leveling Off Point

$$\pi(i) \approx \alpha i^{-k}$$

$$\left| \frac{d\pi(i)}{di} \right| \approx \epsilon \quad \text{for some user-defined tolerance } \epsilon$$

$$i_{level} \approx \left(\frac{k\alpha}{\epsilon} \right)^{1/k+1}$$

Perhaps better:
$$i_{level} \approx f(n) \left(\frac{k\alpha}{\epsilon} \right)^{1/k+1}$$

For WWW:
$$g_{opt} \approx f(n) \left[\frac{2.109\alpha}{\epsilon} \right]^{1/3.109}$$