1. Define the following terms.
   (i) Algebraic and geometric multiplicity of an eigenvalue.
   (ii) Gerschgorin circles and Gerschgorin’s Theorem.
   (iii) Normal matrix.
   (iv) Projection of \( v \in V \) onto \( \mathcal{X} \subseteq V \) along \( \mathcal{Y} \subseteq V \).

2. Prove that \( I_{n \times n} + P^*P \) is nonsingular for every \( k \times n \) matrix \( P \), where \( 1 \leq k \leq n \).

3. Explain why the set \( Z_2 \) of all \( 2 \times 2 \) real matrices having \( \text{trace} \ (A) = 0 \) is a subspace of \( \mathbb{R}^{2 \times 2} \), and then determine \( \dim Z_2 \). (Prove what you claim.)

4. Determine \( \dim N(A) \cap R(B) \) for \( A = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix} \) and \( B = \begin{pmatrix} -1 & 3 & 1 & -4 \\ -2 & -3 & 1 & 0 \\ 2 & 6 & 2 & -8 \end{pmatrix} \).

5. Determine all values of \( \xi \) for which \( A = \begin{pmatrix} \xi & 2 \\ 1 & \xi \\ 1 & 0 \end{pmatrix} \) fails to have an LU factorization.

6. (a) For the \( n \times n \) identity matrix \( I \), prove that \( \|I\| = 1 \) for all induced matrix norms.
    (b) Give an example to show that \( \|I\| = 1 \) is not true for all matrix norms.

7. Using the standard inner product for matrices, compute the Fourier expansion of \( A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \) with respect to the following orthonormal basis for \( \mathbb{R}^{2 \times 2} \).

   \[
   \mathcal{B} = \left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \frac{1}{2} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \right\}
   \]

8. Let \( M = \text{span} \ \{u\} \) for \( u = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \). For \( b = \begin{pmatrix} 4 \\ 8 \end{pmatrix} \), determine the orthogonal projection of \( b \) onto \( M = \text{span} \ \{u\} \), and then determine the orthogonal projection of \( b \) onto \( M^\perp \).

9. Find the singular values of \( A = \begin{pmatrix} \frac{3}{\sqrt{11}} \\ \frac{4}{\sqrt{11}} \end{pmatrix} \).

10. Find all eigenvalues and eigenvectors of \( A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Is \( A \) similar to a diagonal matrix? Explain why.
SOLUTIONS

1. (i) The algebraic multiplicity of \( \lambda \) is the number of times \( \lambda \) is repeated as a root of the characteristic polynomial—i.e., \( \text{alg mult}_A(\lambda) = a_i \) if and only if \((x - \lambda)^{a_1} \cdots (x - \lambda)^{a_i} = 0\) is the characteristic equation for \( A \). The geometric multiplicity of \( \lambda \) is \( \dim N(A - \lambda I) \)—i.e., \( \text{geo mult}_A(\lambda) \) is the maximal number of linearly independent eigenvectors associated with \( \lambda \).

(ii) The Gerschgorin circles that are defined by the rows and columns of \( A \in \mathbb{C}^{n \times n} \) are

\[
|z - a_{ii}| \leq r_i, \text{ where } r_i = \sum_{j \neq i} |a_{ij}| \text{ for } i = 1, 2, \ldots, n.
\]

and

\[
|z - a_{jj}| \leq c_j, \text{ where } c_j = \sum_{i \neq j} |a_{ij}| \text{ for } j = 1, 2, \ldots, n.
\]

Gerschgorin’s theorem says that all eigenvalues of \( A \) are contained in the union \( G_r \) the row circles and also in the union \( G_c \) of the column circles so that \( \sigma(A) \subseteq G_r \cap G_c \).

(iii) A square matrix \( A \) is defined to be normal when \( A^*A = AA^* \), or equivalently, when \( A \) has a full set of orthonormal eigenvectors.

(iv) If \( X = X' \oplus Y' \), then there are unique vectors \( x \in X \) and \( y \in Y \) such that \( v = x + y \). The vector \( x \) is called the projection of \( v \) onto \( X' \) along \( Y' \).

2. Show that \( I + P^*P \) is nonsingular by showing that \( N(I + P^*P) = 0 \). This is done by observing that

\[
x \in N(I + P^*P) \implies x + P^*Px = 0 \implies x^*x + x^*P^*Px = 0 \implies x = 0
\]

because \( x^*x + x^*P^*Px = \|x\|^2 + \|Px\|^2 = 0 \) if and only if \( x = 0 \) and \( Px = 0 \).

3. (a) \( Z_n \) is closed with respect to addition (i.e., \( A, B \in Z_n \implies A + B \in Z_n \)) because if \( \text{trace} (A) = 0 \) and \( \text{trace} (B) = 0 \), then, by linearity of the trace function, \( \text{trace} (A + B) = \text{trace} (A) + \text{trace} (B) = 0 + 0 = 0 \). Similarly, \( Z_n \) is closed with respect to scalar multiplication because if \( \text{trace} (A) = 0 \), then the linearity of trace gives \( \text{trace} (\alpha A) = \alpha \times \text{trace} (A) = 0 \).

(b) A basis for the space of \( 2 \times 2 \) matrices with zero trace is \( \left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \) because every \( 2 \times 2 \) matrix with zero trace is a combination of these three (i.e., they are a spanning set), and this set of three matrices is linearly independent (because \( \alpha_1 B_1 + \alpha_2 B_2 + \alpha_3 B_3 = 0 \implies \alpha_1 = \alpha_2 = \alpha_3 = 0 \)). Therefore, \( \dim Z_2 = 3 \).

4. \( \dim N(A) \cap R(B) = \text{rank} (B) - \text{rank} (AB) = 2 - 1 = 1 \).

5. Looking for zero pivots when reducing by using only type III row operations shows that \( \xi = 0, \pm \sqrt{2}, \pm \sqrt{3} \).

6. (a) \( \|I\|_* = \max_{\|x\|_* = 1} \|I x\|_* = \max_{\|x\|_* = 1} \|x\|_* = 1 \).

(b) It’s false for the Frobenius norm because \( \|I\| = \sqrt{n} \).
7. The Fourier coefficients $\langle U_i | A \rangle = \text{trace}(U_i^T A)$ are

$$\langle U_1 | A \rangle = \frac{2}{\sqrt{2}}, \quad \langle U_2 | A \rangle = 0, \quad \langle U_3 | A \rangle = 1, \quad \langle U_4 | A \rangle = 1,$$

so the Fourier expansion of $A$ is $A = (2/\sqrt{2})U_1 + U_3 + U_4$.

8. $P_M = uu^T/(u^Tu) = (1/10) \begin{pmatrix} 9 & 3 \\ 3 & 1 \end{pmatrix}$, and $P_{M^\perp} = I - P_M = (1/10) \begin{pmatrix} 1 & -3 \\ -3 & 9 \end{pmatrix}$, so $P_M b = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$, and $P_{M^\perp} b = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$.

9. In general, the singular values are the positive square roots of the eigenvalues of $A^T A$, so in this case there is only one singular value, and it is $\sigma = \sqrt{36} = 6$.

10. $0 = \det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ 0 & -\lambda \end{pmatrix} = \lambda^2$ means that $\lambda = 0$ is an eigenvalue of algebraic multiplicity two. The eigenvectors are solutions to $Ax = 0$, which are $x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, where $x_1$ is a free variable. $A$ is not similar to a diagonal matrix because there is not a set of two linearly independent eigenvectors, or equivalently, because $2 = \text{alg mult}(0) \neq \text{geo mult}(0) = 1$. 