Reordering and Updating for the PageRank Problem

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Outline

- PageRank Solution Methods
- A Reordering for PageRank
- Updating PageRank
Google

Indexing

- Must index key terms on each page
  Robots crawl the web — software does indexing

- Inverted file structure  (like book index: terms $\rightarrow$ to pages)
  
  $Term_1 \rightarrow P_i, P_j, \ldots$
  $Term_2 \rightarrow P_k, P_l, \ldots$
  
  

Ranking

- Determine a “PageRank” for each page $P_i, P_j, P_k, P_l, \ldots$
  Query independent — Based only on link structure

- Query matching
  
  $Q = Term_1, Term_2, \ldots$ produces $P_i, P_j, P_k, P_l, \ldots$

- Return $P_i, P_j, P_k, P_l, \ldots$ to user in order of PageRank
Google’s PageRank Idea

(Sergey Brin & Lawrence Page 1998)

- Rankings are not query dependent
  Depend only on link structure
  Off-line calculations

- Your page $P$ has some rank $r(P)$

- Adjust $r(P)$ higher or lower depending on ranks of pages that point to $P$

- Importance is not number of in-links or out-links
  One link to $P$ from Yahoo! is important
  Many links to $P$ from me is not

- Yahoo! points many places — value of link to $P$ is diluted
PageRank

The Definition

\[ r(P) = \sum_{P \in \mathcal{B}_P} \frac{r(P)}{|P|} \]

\[ \mathcal{B}_P = \{\text{all pages pointing to } P\} \]

\[ |P| = \text{number of out links from } P \]

Successive Refinement

Start with \( r_0(P_i) = \frac{1}{n} \) for all pages \( P_1, P_2, \ldots, P_n \)

Iteratively refine rankings for each page

\[ r_1(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_0(P)}{|P|} \]

\[ r_2(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_1(P)}{|P|} \]

\[ \vdots \]

\[ r_{j+1}(P_i) = \sum_{P \in \mathcal{B}_{P_i}} \frac{r_j(P)}{|P|} \]
In Matrix Notation

After Step $j$

$$\pi_j^T = [r_j(P_1), r_j(P_2), \ldots, r_j(P_n)]$$

$$\pi_{j+1}^T = \pi_j^T P \quad \text{where} \quad p_{ij} = \begin{cases} 1/|P_i| & \text{if } i \rightarrow j \\ 0 & \text{otherwise} \end{cases}$$

PageRank = \lim_{j \to \infty} \pi_j^T = \pi^T \quad \text{(provided limit exists)}

It’s Almost a Markov Chain

$P$ has row sums = 1 for ND nodes, row sums = 0 for D nodes
In Matrix Notation

After Step $j$

$$\pi^T_j = [r_j(P_1), r_j(P_2), \cdots, r_j(P_n)]$$

$$\pi^T_{j+1} = \pi^T_j P \quad \text{where} \quad p_{ij} = \begin{cases} 1/|P_i| & \text{if } i \to j \\ 0 & \text{otherwise} \end{cases}$$

PageRank $= \lim_{j \to \infty} \pi^T_j = \pi^T$ 

(provided limit exists)

It’s Almost a Markov Chain

$P$ has row sums $= 1$ for ND nodes, row sums $= 0$ for D nodes

Stochasticity Fix: $\tilde{P} = P + av^T$. 

($a_i=1$ for $i \in D$, 0, o.w.)
In Matrix Notation

After Step \( j \)

\[
\pi_j^T = \left[ r_j(P_1), r_j(P_2), \cdots, r_j(P_n) \right]
\]

\[
\pi_{j+1}^T = \pi_j^T P \quad \text{where} \quad p_{ij} = \begin{cases} 
1/|P_i| & \text{if } i \rightarrow j \\
0 & \text{otherwise}
\end{cases}
\]

PageRank = \( \lim_{j \to \infty} \pi_j^T = \pi^T \) (provided limit exists)

It’s Almost a Markov Chain

\( P \) has row sums = 1 for ND nodes, row sums = 0 for D nodes

Stochasticity Fix: \( \tilde{P} = P + av^T \). \( (a_i=1 \text{ for } i \in D, 0, \text{ o.w.}) \)

Each \( \pi_j^T \) is a probability distribution vector \( \left( \sum_i r_j(P_i)=1 \right) \)

\[
\pi_{j+1}^T = \pi_j^T \tilde{P} \quad \text{is random walk on the graph defined by links}
\]

\( \pi^T = \lim_{j \to \infty} \pi_j^T = \text{stationary probability distribution} \)
Random Surfer

Web Surfer Randomly Clicks On Links

Long-run proportion of time on page $P_i$ is $\pi_i$

Problems

Dead end page (nothing to click on) $(\pi^T$ not well defined)

Could get trapped into a cycle $(P_i \rightarrow P_j \rightarrow P_i)$ (No convergence)

Convergence

Markov chain must be irreducible and aperiodic

Bored Surfer Enters Random URL

Irreducibility Fix:

$$\tilde{P} = \alpha \tilde{P} + (1 - \alpha)E \quad e_{ij} = 1/n \quad \alpha \approx 0.85$$

$$\tilde{P} = \alpha P + \alpha a v^T + (1 - \alpha)E$$

Different $E = ev^T$ and $\alpha$ allow customization & speedup, yet rank-one update maintained; $\tilde{P} = \alpha P + (\alpha a + (1 - \alpha) e)v^T$
Computing $\pi^T$

A Big Problem

Solve $\pi^T = \pi^T \tilde{P}$

$\pi^T (I - \tilde{P}) = 0$

(stationary distribution vector)

(too big for direct solves)
Computing $\pi^T$

**A Big Problem**

Solve $\pi^T = \pi^T \tilde{P}$  \hspace{3cm} (stationary distribution vector)

$\pi^T (I - \tilde{P}) = 0$  \hspace{3cm} (too big for direct solves)

Start with $\pi^T_0 = e/n$ and iterate $\pi^T_{j+1} = \pi^T_j \tilde{P}$  \hspace{3cm} (power method)
Power Method to compute PageRank

\[ \pi_0^T = \mathbf{e}^T / n \]

until convergence, do

\[ \pi_{j+1}^T = \pi_j^T \tilde{P} \quad \text{(dense computation)} \]

end
Power Method to compute PageRank

\[ \pi_0^T = e^T / n \]

until convergence, do

\[ X \pi_{j+1}^T = \pi_j^T \tilde{P} \]

• \[ \pi_{j+1}^T = \alpha \pi_j^T \tilde{P} + (1 - \alpha) \pi_j^T e v^T \]

end

(dense computation)

(sparser computation)
Power Method to compute PageRank

\[ \pi_0^T = \frac{e^T}{n} \]

until convergence, do

\[ X \pi_{j+1}^T = \pi_j^T \tilde{P} \]  
(dense computation)

\[ X \pi_{j+1}^T = \alpha \pi_j^T \tilde{P} + (1 - \alpha) \pi_j^T e v^T \]  
(sparser computation)

\[ \pi_{j+1}^T = \alpha \pi_j^T P + (\alpha \pi_j^T a + (1 - \alpha)) v^T \]  
(even less computation)

end

• \( P \) is very, very sparse with about 3-10 nonzeros per row.

• \( \Rightarrow \) one vector-matrix mult. is \( O(nnz(P)) \approx O(n). \)
Convergence

Can prove $\lambda_2(\bar{P}) = \alpha$

($\Rightarrow$ asymptotic rate of convergence of PageRank method is rate at which $\alpha^k \rightarrow 0$)

Google

- uses $\alpha = .85$
- report 50-100 iterations til convergence
- still takes days to converge
Enhancements to the PR power method

- Kamvar et al. Extrapolation
- Kamvar et al. Adaptive PageRank
- Kamvar et al. BlockRank
- Lee et al. Lumpability of Dangling Nodes
- Langville/Meyer: Updating PageRank
- Ipsen/Kirkland: more theory for Langville/Meyer
Linear System Formulation

For \( \tilde{P} \)

\[
\pi^T(I - \tilde{P}) = 0^T \quad \text{and} \quad \pi^T e = 1.
\]

For \( P \)

\[
\pi^T(I - \alpha \tilde{P}) = (1 - \alpha)v^T \quad \text{and} \quad \pi^T e = 1.
\]

For \( P \)

\[
\pi^T(I - \alpha P) = v^T \quad \text{and} \quad \pi^T e = 1.
\]

\( P \) is very sparse, 3-10 nonzeros per row
Properties of \((I - \alpha P)\):

1. \((I - \alpha P)\) is nonsingular.

2. \((I - \alpha P)\) is an \(M\)-matrix.

3. The row sums of \((I - \alpha P)\) are either \(1 - \alpha\) for ND nodes or 1 for D nodes.

4. \(\|I - \alpha P\|_\infty = 1 + \alpha\).

5. Since \((I - \alpha P)\) is an \(M\)-matrix, \((I - \alpha P)^{-1} \geq 0\).

6. The row sums of \((I - \alpha P)^{-1}\) are equal to 1 for the D nodes and less than or equal to \(1/(1 - \alpha)\) for the ND nodes.

7. The condition number \(\kappa_\infty(I - \alpha P) \leq (1 + \alpha)/(1 - \alpha)\).

8. The row of \((I - \alpha P)^{-1}\) corresponding to D node \(i\) is \(e_i^T\).
ND-D Reordering

\[ P = \begin{bmatrix} ND & D \\ ND & P_{11} & P_{12} \\ 0 & 0 & 0 \end{bmatrix}. \]

\[ (I - \alpha P) = \begin{bmatrix} I - \alpha P_{11} & -\alpha P_{12} \\ 0 & I \end{bmatrix}. \]

\[ (I - \alpha P)^{-1} = \begin{bmatrix} (I - \alpha P_{11})^{-1} & \alpha(I - \alpha P_{11})^{-1}P_{12} \\ 0 & I \end{bmatrix}. \]
Algorithm 1: ND-D Reordering

Solve $\pi^T(I - \alpha P) = v^T$ and $\pi^T e = 1$.

Algorithm 1:

1. Solve for $\pi^{T}_1$ in $\pi^{T}_1(I - \alpha P_{11}) = v^{T}_1$.

2. Compute $\pi^{T}_2 = \alpha \pi^{T}_1 P_{12} + v^{T}_2$.

3. Normalize $\pi^T = [\pi^T_1 \pi^T_2]/\|[\pi^T_1 \pi^T_2]\|_1$.

Pro: one small system solve, plus forward substitution.

Analog: Lee et al. lumpable D node Markov formulation.
Extension of ND-D Reordering

- Continue locating 0 rows in submatrices of \((I - \alpha P)\) until no 0 rows remain. Amounts to a reordering of indices.

Before Reordering

After Reordering
Algorithm 2: Recursive ND-D Reordering

Solve $\pi^T(I - \alpha P) = v^T$ and $\pi^T e = 1$.

Algorithm 2:

1. Reorder the states of the original Markov chain, so that the reordered matrix has the 0 block structure. $O(\text{nnz}(P)) \approx 1$ power iter.

2. Solve for $\pi_1^T$ in $\pi_1^T(I - \alpha P_{11}) = v_1^T$. Jacobi method with rate of conv. $\leq \alpha$

3. Compute $\pi_2^T = \alpha \pi_1^T P_{12} + v_2^T$.

4. Compute $\pi_3^T = \alpha \pi_1^T P_{13} + \alpha \pi_2^T P_{23} + v_3^T$.

5. Compute $\pi_b^T = \alpha \pi_1^T P_{1b} + \alpha \pi_2^T P_{2b} + \cdots + \alpha \pi_{b-1}^T P_{b-1,b} + v_b^T$. $O(\text{nnz}(P))$

6. Normalize $\pi^T = [\pi_1^T \pi_2^T \cdots \pi_b^T]/\| [\pi_1^T \pi_2^T \cdots \pi_b^T]\|_1$.

Pro: even smaller system solve, plus forward substitution.

Speedup: by factor of $\text{nnz}(P)/\text{nnz}(P_{11})$ (estimated)
### Results of Reordered PageRank

<table>
<thead>
<tr>
<th></th>
<th>EPA.dat</th>
<th>CA.dat</th>
<th>NCS.dat</th>
<th>ND.dat</th>
<th>SU450k.dat</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>PR</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Time</strong></td>
<td>3.80</td>
<td>6.63</td>
<td>13.17</td>
<td>177.16</td>
<td>237.37</td>
</tr>
<tr>
<td><strong>Iter.</strong></td>
<td>159</td>
<td>176</td>
<td>162</td>
<td>166</td>
<td>164</td>
</tr>
<tr>
<td>(n(P))</td>
<td>5,042</td>
<td>9,664</td>
<td>10,000</td>
<td>325,729</td>
<td>451,237</td>
</tr>
<tr>
<td>(nz(P))</td>
<td>9,563</td>
<td>16,873</td>
<td>101,118</td>
<td>1,497,134</td>
<td>1,082,604</td>
</tr>
<tr>
<td><strong>RePR</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Time</strong></td>
<td>.59</td>
<td>1.42</td>
<td>7.65</td>
<td>130.54</td>
<td>52.84</td>
</tr>
<tr>
<td><strong>Iter.</strong></td>
<td>155</td>
<td>169</td>
<td>160</td>
<td>170</td>
<td>145</td>
</tr>
<tr>
<td>(b)</td>
<td>10</td>
<td>9</td>
<td>5</td>
<td>18</td>
<td>12</td>
</tr>
<tr>
<td>(n(P_{11}))</td>
<td>704</td>
<td>2,622</td>
<td>7,136</td>
<td>127,472</td>
<td>84,861</td>
</tr>
<tr>
<td>(nz(P_{11}))</td>
<td>1,330</td>
<td>5,238</td>
<td>79,230</td>
<td>1,191,761</td>
<td>267,566</td>
</tr>
<tr>
<td><strong>Speed</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td><strong>Est.</strong></td>
<td>7.2</td>
<td>3.2</td>
<td>1.3</td>
<td>1.3</td>
<td>4.0</td>
</tr>
<tr>
<td><strong>Act.</strong></td>
<td>6.4</td>
<td>4.7</td>
<td>1.7</td>
<td>1.4</td>
<td>4.5</td>
</tr>
</tbody>
</table>

- can do no worse than original PR power method
- Speedup is dataset-dependent
Langville/Meyer Updating

Motivation

– Updating PR is huge problem. Currently done monthly, but web changes hourly.
– Chien et al. use aggregation to focus on pages whose PR is most likely to change.

Idea

– Use iterative aggregation to extend Chien idea.
– Focus on bad states, aggregate good, fast-converging states into one superstate.
– ⇒ only work on much smaller aggregated chain.

Results

– speedup by factor of 5-10 on some datasets.

Issue

– Partitioning into good and bad states is hard, and IAD is very sensitive to partition.
Idea behind Aggregation

Best for NCD systems

(Simon and Ando (1960s), Courtois (1970s))

\[
A = \begin{bmatrix} c_1 & c_2 & c_3 \\
\end{bmatrix}
\]

\[
a_{ij} = \Pi^T P_{ij} e^T
\]

\[
\Pi^T \approx \begin{bmatrix} \xi_1 \Pi^T & \xi_2 \Pi^T & \xi_3 \Pi^T \\
\end{bmatrix}
\]

Pro

exploits structure to reduce work

Con

produces an approximation, quality is dependent on degree of coupling
Iterative Aggregation

- Problem: repeated aggregation leads to fixed point.
- Solution: Do a power step to move off fixed point.
- Do this iteratively. Approximations improve and approach exact solution.
- Success with NCD systems, not in general.

Input: approximation to $\Pi^T$
get censored distributions $\Pi^T \Pi^T \Pi^T$  
get coupling constants $\xi_i$
Output: get approximate global stationary distribution $\Pi^T = \begin{bmatrix} \xi_1 \Pi^T & \xi_2 \Pi^T & \xi_3 \Pi^T \end{bmatrix}$
Output: move off fixed point with power step
Exact Aggregation

(Meyer 1989)

$$\mathbf{P} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ \end{bmatrix}$$

$$\mathbf{A} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \mathbf{c}_3 \\ \end{bmatrix}$$

$$a_{ij} = s_i^T P_{ij} e$$

$$\mathbf{\xi}^T = \left[ \begin{array}{c} \xi_1 \\ \xi_2 \\ \xi_3 \end{array} \right]$$

$$\mathbf{\Pi}^T = \left[ \begin{array}{ccc} \xi_1 s_1^T & \xi_2 s_2^T & \xi_3 s_3^T \end{array} \right]$$

Pro

only one step needed to produce exact global vector

Con

SC matrices $\mathbf{S}_i$ are very expensive to compute

$s_i^T =$ censored (stat.) dist. of stochastic complement $\mathbf{S}_i$

$$S_i = P_{i1} + P_{i*} (I - P_*)^{-1} \mathbf{P}_{*i}$$

For 2-level partition,

$$S_i = P_{i1} + P_{i2} (I - P_{22})^{-1} \mathbf{P}_{21}$$
### Aggregation

#### Partitioned Matrix

\[ \mathbf{P}_{n \times n} = \frac{G}{G} \left( \begin{array}{cc} \mathbf{G} & \mathbf{G} \\ \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{array} \right) = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & \mathbf{P}_{22} \end{bmatrix} \]

\[ \pi^T = (\pi_1, \ldots, \pi_g \mid \pi_{g+1}, \ldots, \pi_n) \]

#### Advantages of this Partition

- \(p_{11} \cdots p_{gg}\) are \(1 \times 1\) \(\implies\) Stochastic complements = 1

- \(\implies\) censored distributions = 1

Only one significant complement

\[ \mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12} \]

Only one significant censored dist

\[ \mathbf{s}_2^T \mathbf{S}_2 = \mathbf{s}_2^T \]

A/D Theorem \(\implies\)

\[ \mathbf{s}_2^T = (\pi_{g+1}, \ldots, \pi_n) / \sum_{i=g+1}^n \pi_i \]
### Aggregation Matrix

\[
A = \begin{bmatrix}
  p_{11} & \cdots & p_{1g} & r^T_{1e} \\
  \vdots & \ddots & \vdots & \vdots \\
  p_{g1} & \cdots & p_{gg} & r^T_{ge} \\
  s^T_2 c_1 & \cdots & s^T_2 c_g & s^T_2 P_{22} e
\end{bmatrix}
= \begin{bmatrix}
P_{11} & P_{12} e \\
s^T_2 P_{21} & 1 - s^T_2 P_{21} e
\end{bmatrix}_{(g+1) \times (g+1)}
\]

### The Aggregation/Disaggregation Theorem

If \( \alpha^T = (\alpha_1, \ldots, \alpha_g, \alpha_{g+1}) = \text{stationary dist for } A \)

Then \( \pi^T = (\alpha_1, \ldots, \alpha_g \mid \alpha_{g+1}s^T_2) = \text{stationary dist for } P \)

### Trouble! Always A Big Problem

\( G \) small \( \Rightarrow \) \( G \) big \( \Rightarrow \) \( S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12} \) large

\( G \) big \( \Rightarrow \) \( A \) large
Approximate Aggregation

Assumption

Updating involves relatively few states

\[ G \text{ small } \Rightarrow A = \begin{bmatrix} P_{11} & P_{12}e \\ s_2^T P_{21} & 1 - s_2^T P_{21}e \end{bmatrix} \text{ small} \]

Approximation

\[ (\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n), \]

where \( \phi^T \) is old PageRank vector and \( \pi^T \) is new, updated PageRank

\[ s_2^T = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^{n} \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^{n} \phi_i} = \tilde{s}_2^T \]

(avoiding computing \( \tilde{s}_2^T \) for large \( S_2 \))

\[ A \approx \tilde{A} = \begin{bmatrix} P_{11} & P_{12}e \\ \tilde{s}_2^T P_{21} & 1 - \tilde{s}_2^T P_{21}e \end{bmatrix} \]

\[ \alpha^T \approx \tilde{\alpha}^T = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1}) \]

\[ \pi^T \approx \tilde{\pi}^T = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_g \mid \tilde{\alpha}_{g+1} \tilde{s}_2^T) \] (not bad)
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO

Can’t do A/D twice — a fixed point emerges

Solution

Perturb A/D output to move off of fixed point
Move it in direction of solution

\[ \tilde{\pi}^T = \tilde{\pi}^T P \]  
(a smoothing step)

The Iterative A/D Updating Algorithm

Determine the “G-set” partition \( S = G \cup \overline{G} \)
Approximate A/D step generates approximation \( \tilde{\pi}^T \)
Smooth the result \( \tilde{\tilde{\pi}}^T = \tilde{\pi}^T P \)
Use \( \tilde{\pi}^T \) as input to another approximate aggregation step

: :
How to Partition for Updating Problem?

Intuition

- There are some bad states ($G$) and some good states ($\overline{G}$).
- Give more attention to bad states. Each state in $G$ forms a partitioning level. Much progress toward correct PageRank is made during aggregation step.
- Lump good states in $\overline{G}$ into 1 superstate. Progress toward correct PageRank is made during smoothing step (power iteration).
Definitions for “Good” and “Bad”

1. Good = states least likely to have $\pi_i$ change
   Bad = states most likely to have $\pi_i$ change

2. Good = states with smallest $\pi_i$ after $k$ transient steps
   Bad = states “nearby”, with largest $\pi_i$ after $k$ transient steps

3. Good = smallest $\pi_i$ from old PageRank vector
   Bad = largest $\pi_i$ from old PageRank vector

4. Good = fast–converging states
   Bad = slow–converging states
Determining “Fast” and “Slow”

Consider power method and its rate of convergence

\[ \pi_{k+1}^T = \pi_k^T P = \pi_k^T e \pi^T + \lambda_2^k \pi_k^T x_2 y_2^T + \lambda_3^k \pi_k^T x_3 y_3^T + \cdots + \lambda_n^k \pi_k^T x_n y_n^T \]

Asymptotic rate of convergence is rate at which \( \lambda_2^k \rightarrow 0 \)

Consider convergence of elements

Some states converge to stationary value faster than \( \lambda_2 \)–rate, due to LH e–vector \( y_2^T \).

Partitioning Rule

Put states with largest \( |y_2^T|_i \) values in bad group \( G \), where they receive more individual attention in aggregation method.

Practicality

\( y_2^T \) expensive, but for PageRank problem, Kamvar et al. show states with large \( \pi_i \) are slow-converging. \( \Rightarrow \) inexpensive soln = use old \( \pi^T \) to determine \( G \). (adaptively approximate \( y_2^T \))
Implications of Web’s scale-free nature

Facts:

(1) \( \pi^T \) follows power law since WWW is scale-free

(2) not all pages converge to their PageRanks at same rate

(3) pages with high PR are slow-converging

\[ \Rightarrow \text{very few pages are slow-converging, but these are the pages that cause power method to drag on} \]
Power law for PageRank

Scale-free Model of Web network creates power laws

(Kamvar, Barabasi, Raghavan)
Convergence

**Theorem**

Always converges to stationary dist \( \pi^T \) for \( P \)

Converges for all partitions \( S = G \cup \overline{G} \)

Rate of convergence is rate at which \( S^n_2 \) converges

\[ S_2 = P_{22} + P_{21}(I-P_{11})^{-1}P_{12} \]

Dictated by Jordan structure of \( \lambda_2(S_2) \)

\[ \lambda_2(S_2) \text{ simple} \implies \pi_k^T \to \pi^T \] at the rate at which \( \lambda_2^n \to 0 \)

**The Game**

Goal now is to find a relatively small \( G \) that minimizes \( \lambda_2(S_2) \)
Motivation

– L/M prove updating method converges at rate \((\lambda_2(S_2))^k \rightarrow 0\).
– Ipsen/Kirkland wonder: can \(\lambda_2(S_2) > \alpha\) ?

Results

– \(\lambda_2(S_2) \leq \alpha\) for all partitions.
– \(\lambda_2(S_2) < \alpha\) under two trivial assumptions on \(P\).
  
  (\(P\) is reducible, and at least one page in each essential class does not self-link)
Ipsen/Kirkland Updating Theory

Motivation

- L/M prove updating method converges at rate \((\lambda_2(S_2))^k \rightarrow 0\).
- Ipsen/Kirkland wonder: can \(\lambda_2(S_2) > \alpha\) ?

Results

- \(\lambda_2(S_2) \leq \alpha\) for all partitions.
- \(\lambda_2(S_2) < \alpha\) under two trivial assumptions on \(P\).
  
  \((P\) is reducible, and at least one page in each essential class does not self-link) 

But ... how do we find partition so that \(\lambda_2(S_2) << \alpha\) ?
Experiments

Test Networks From Crawl Of Web

NCState  (NCSU internal crawl)
10,000 nodes  101,118 links

California  (Sites concerning “california” query)
9,664 nodes  16,150 links
Parameters

Number Of Nodes (States) Added

50

Number Of Nodes (States) Removed

30

Number Of Links Added

300

(Different values have little effect on results)

Number Of Links Removed

200

Stopping Criterion

1-norm of residual $< 10^{-10}$
## NC State

### Power Method

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Time</th>
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<tbody>
<tr>
<td>162</td>
<td>9.79</td>
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### Iterative Aggregation

| \(|G|\)   | Iterations | Time  |
|-----------|-------------|-------|
| 500       | 160         | 10.18 |
| 1000      | 51          | 3.92  |
| 1500      | 33          | 2.82  |
| 2500      | 16          | 2.15  |
| 3000      | 13          | 1.99  |
| 5000      | 7           | 1.77  |

*nodes* = 10,000  
*links* = 101,118
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| **2000** | **21**   | **2.22** |
| 2500| 16         | 2.15   |
| 3000| 13         | 1.99   |
| 5000| 7          | 1.77   |

$nodes = 10,000 \quad links = 101,118$
### California

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<td>5000</td>
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nodes = 9,664  links = 16,150
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*nodes = 9,664  links = 16,150*
Advantage

— updating algorithm can be combined with other PR acceleration methods.

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nodes = 10,000   links = 101,118
Residual Plot for NC State

- Power method
- Power method with quad. extrap.
- Iterative aggregation
- Iterative aggregation with quad. extrap.
Large-Scale Implementation

Partitioning
   — need more theoretical work on good partitioning.

IAD’s Aggregated System Solve
   — direct vs. sparse methods

Simulating updates to Web
   — how to do this accurately, and keep scale-free properties of web
   — need collections of the web over time.
Conclusions

- An appropriate reordering of the pages of the web can greatly speed the PageRank computation.

- Aggregation methods reduce PageRank computation for the updating problem. However, partitioning is a difficult, unresolved issue.

- Many of these methods can be combined to achieve even greater speedups.

- We are moving closer to lofty goal of computing real-time personalized PageRank.