Updating The PageRank Vector

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The PageRank Vector

Definition

\[ \pi^T = \text{stationary distribution of a Markov chain} \]

\[ P = tT + (1 - t)E \quad 0 < t < 1 \]

Irreducible & Aperiodic
The PageRank Vector

Definition

\[ \pi^T = \text{stationary distribution of a Markov chain} \]

\[ P = tT + (1 - t)E \quad 0 < t < 1 \]

Irreducible & Aperiodic

Big Eigenvector Problem

Solve \[ \pi^T = \pi^T P \quad \pi^T e = 1 \]

\[ n = O(10^9) \]

“World’s Largest Matrix Computation” (Cleve Moler)
Computing $\pi^T$

**Iterate**

Start with $\pi^T_0 = e/n$ and iterate $\pi^T_{j+1} = \pi^T_j P$ (power method)

**Convergence Time**

Use to be measured in days
Computing $\pi^T$

Iterate

Start with $\pi^T_0 = e/n$ and iterate $\pi^T_{j+1} = \pi^T_j P$ (power method)

Convergence Time

Use to be measured in days

Now ???

Recent Advances

Extrapolation methods for accelerating PageRank, Kamvar, Haveliwala, Manning, Golub, 03
Exploiting the block structure of the web for computing PageRank, K, H, M, Golub, 03
Adaptive methods for the computation of PageRank, Kamvar, Haveliwala, Golub, 03
Partial state space aggregation based on lumpability and its application to PageRank,

Chris Lee, 03
Updating

Easy Problem

No pages added — No pages removed

— Size does not change — only probabilities change
Updating

Easy Problem

No pages added — No pages removed

- Size does not change — only probabilities change

Hard Problem

Both pages & links are added or removed

- Both size & probabilities change
Updating

Easy Problem

No pages added — No pages removed

— Size does not change — only probabilities change

Hard Problem

Both pages & links are added or removed

— Both size & probabilities change

The Trouble

Prior results are not much help

— Google just restarts from scratch every few weeks
Perron Complementation

Perron Frobenius

\( P \geq 0 \) irreducible \( \implies \rho = \rho(P) \) simple eigenvalue

Unique Left-Hand Perron Vector

\( \pi^T P = \rho \pi^T \quad \pi^T > 0 \quad \|\pi^T\|_1 = 1 \)
Perron Complementation

**Perron Frobenius**

\[ P \geq 0 \text{ irreducible } \implies \rho = \rho(P) \text{ simple eigenvalue} \]

**Unique Left-Hand Perron Vector**

\[ \pi^T P = \rho \pi^T \quad \pi^T > 0 \quad \|\pi^T\|_1 = 1 \]

**Partition**

\[ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \]

Shift \( P \) by \( \rho \) \quad \text{Schur Complements} \quad \text{Shift back by} \ \rho
Perron Complementation

**Perron Frobenius**

\[ \mathbf{P} \geq 0 \quad \text{irreducible} \quad \Rightarrow \quad \rho = \rho(\mathbf{P}) \quad \text{simple eigenvalue} \]

**Unique Left-Hand Perron Vector**

\[ \pi^T \mathbf{P} = \rho \pi^T \quad \pi^T > 0 \quad \| \pi^T \|_1 = 1 \]

**Partition**

\[ \mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \]

Shift \( \mathbf{P} \) by \( \rho \) \quad \rightarrow \quad \text{Schur Complements} \quad \rightarrow \quad \text{Shift back by} \ \rho

**Perron Complements**

\[ S_1 = \mathbf{P}_{11} + \mathbf{P}_{12}(\rho \mathbf{I} - \mathbf{P}_{22})^{-1}\mathbf{P}_{21} \]

\[ S_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\rho \mathbf{I} - \mathbf{P}_{11})^{-1}\mathbf{P}_{12} \]
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

$S_i \geq 0$
Inherited Properties

For \( P \geq 0 \) irreducible with \( \rho = \rho(P) \)

\[ S_i \geq 0 \]

\( S_i \) is irreducible
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

$S_i \geq 0$

$S_i$ is irreducible

$\rho(S_i) = \rho(P) = \rho$
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

$S_i \geq 0$

$S_i$ is irreducible

$\rho(S_i) = \rho(P) = \rho$

For $P$ stochastic

$S_i$ is stochastic

$S_i$ represents a censored Markov chain
Inherited Properties

For $P \geq 0$ irreducible with $\rho = \rho(P)$

- $S_i \geq 0$
- $S_i$ is irreducible
- $\rho(S_i) = \rho(P) = \rho$

For $P$ stochastic

- $S_i$ is stochastic
- $S_i$ represents a censored Markov chain

Censored Perron vectors

- $s_i^T = \text{Left-hand Perron vector for } S_i$
- $s_i^T S_i = \rho \ s_i^T$
Aggregation

Objective

Use $s_1^T s_2^T \cdots$ to build $\pi^T$
Aggregation

Objective

Use $s_1^T s_2^T \cdots$ to build $\pi^T$

Aggregation Matrix

$$A = \begin{bmatrix} s_1^T P_{11} e & s_1^T P_{12} e \\ s_2^T P_{21} e & s_2^T P_{22} e \end{bmatrix}_{2 \times 2}$$
Aggregation

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$A \geq 0$

$A$ is irreducible
Aggregation

Objective

Use \( s_1^T s_2^T \cdots \) to build \( \pi^T \)

Aggregation Matrix

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A = \begin{bmatrix}
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s_2^T P_{21} e & s_2^T P_{22} e
\end{bmatrix}_{2 \times 2}
\]

Inherited Properties

\( A \geq 0 \)

\( A \) is irreducible

\( \rho(A) = \rho = \rho(P) = \rho(S_i) \)
Aggregation

Objective

Use $s_1^T s_2^T \cdots$ to build $\pi^T$

Aggregation Matrix

$$A = \begin{bmatrix} s_1^T P_{11} e & s_1^T P_{12} e \\ s_2^T P_{21} e & s_2^T P_{22} e \end{bmatrix}_{2 \times 2}$$

Inherited Properties

$A \geq 0$

$A$ is irreducible

$\rho(A) = \rho = \rho(P) = \rho(S_i)$

$P$ stochastic $\implies A$ stochastic
Disaggregation

The A / D Theorem

If

\[ s_i^T = \text{Perron vectors for } S_i = P_{ii} + P_{i*}(\rho I - P_{**})^{-1}P_{i*} \]

\[ \alpha^T = (\alpha_1, \alpha_2) = \text{Perron vector for } A = \begin{bmatrix} s_1^T P_{11}e & s_1^T P_{12}e \\ s_2^T P_{21}e & s_2^T P_{22}e \end{bmatrix}_{2 \times 2} \]

then

\[ \pi^T = (\alpha_1 s_1^T \mid \alpha_2 s_2^T) = \text{Perron vector for } P_{n \times n} \]
Disaggregation

The A / D Theorem

If

\[ \mathbf{s}_i^T = \text{Perron vectors for } \mathbf{S}_i = \mathbf{P}_{ii} + \mathbf{P}_{i*}(\rho \mathbf{I} - \mathbf{P}_{**})^{-1}\mathbf{P}_{i*} \]

\[ \alpha^T = (\alpha_1, \alpha_2) = \text{Perron vector for } \mathbf{A} = \begin{bmatrix} \mathbf{s}_1^T \mathbf{P}_{11} \mathbf{e} & \mathbf{s}_1^T \mathbf{P}_{12} \mathbf{e} \\ \mathbf{s}_2^T \mathbf{P}_{21} \mathbf{e} & \mathbf{s}_2^T \mathbf{P}_{22} \mathbf{e} \end{bmatrix}_{2\times2} \]

then

\[ \pi^T = (\alpha_1 \mathbf{s}_1^T | \alpha_2 \mathbf{s}_2^T) = \text{Perron vector for } \mathbf{P}_{n \times n} \]

Corollary

\[ \mathbf{s}_1^T = (\pi_1, \ldots, \pi_g)/\sum_{i=1}^g \pi_i \]

\[ \mathbf{s}_2^T = (\pi_{g+1}, \ldots, \pi_n)/\sum_{i=g+1}^n \pi_i \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix} \]  
\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \]
Updating By Aggregation

**Prior Data**

\[ Q_{m \times m} = \text{Old Google Matrix} \quad \text{(known)} \]

\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \quad \text{(known)} \]

**Updated Data**

\[ P_{n \times n} = \text{New Google Matrix} \quad \text{(known)} \]

\[ \pi^T = (\pi_1, \pi_2, \ldots, \pi_n) = \text{New PageRank Vector} \quad \text{(unknown)} \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix} \quad (\text{known}) \]

\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector} \quad (\text{known}) \]

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\[ P_{n \times n} = \text{New Google Matrix} \quad (\text{known}) \]

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Separate Pages Likely To Be Most Affected

\[ G = \{\text{most affected}\} \quad \overline{G} = \{\text{less affected}\} \quad S = G \cup \overline{G} \]
Updating By Aggregation

Prior Data

\[ Q_{m \times m} = \text{Old Google Matrix (known)} \]

\[ \phi^T = (\phi_1, \phi_2, \ldots, \phi_m) = \text{Old PageRank Vector (known)} \]

Updated Data

\[ P_{n \times n} = \text{New Google Matrix (known)} \]

\[ \pi^T = (\pi_1, \pi_2, \ldots, \pi_n) = \text{New PageRank Vector (unknown)} \]

Separate Pages Likely To Be Most Affected

\[ G = \{\text{most affected}\} \quad \overline{G} = \{\text{less affected}\} \quad S = G \cup \overline{G} \]

New pages (and neighbors) go into \( G \)
Aggregation

Partition

\[ P_{n \times n} = \frac{G}{\overline{G}} \begin{pmatrix} G & \overline{G} \\ P_{11} & P_{12} \\ \overline{P}_{21} & P_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & P_{22} \end{bmatrix} \]

\[ \pi^T = (\pi_1, \ldots, \pi_g \mid \pi_{g+1}, \ldots, \pi_n) \]
Aggregation

Partition

\[
\mathbf{P}_{n \times n} = \frac{G}{\overline{G}} \begin{pmatrix}
G & \overline{G} \\
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix} = \begin{bmatrix}
p_{11} & \cdots & p_{1g} & r_1^T \\
\vdots & \ddots & \vdots & \vdots \\
p_{g1} & \cdots & p_{gg} & r_g^T \\
c_1 & \cdots & c_g & P_{22}
\end{bmatrix}
\]

\[
\pi^T = (\pi_1, \ldots, \pi_g \mid \pi_{g+1}, \ldots, \pi_n)
\]

Perron Complements

\[p_{11} \cdots p_{gg} \text{ are } 1 \times 1 \implies \text{Perron complements} = 1\]

\[\implies \text{Perron vectors} = 1\]
Aggregation

Partition

\[ P_{n \times n} = \begin{bmatrix} \frac{G}{G} P_{11} & \frac{G}{G} P_{12} \\ \frac{G}{G} P_{21} & \frac{G}{G} P_{22} \end{bmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & P_{22} \end{bmatrix} \]

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Perron Complements

\[ p_{11} \cdots p_{gg} \text{ are } 1 \times 1 \quad \Rightarrow \quad \text{Perron complements } = 1 \]

\[ \quad \Rightarrow \text{Perron vectors } = 1 \]

One significant complement \[ S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12} \]
Aggregation

Partition

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P_{n \times n} = \frac{G}{G} \begin{pmatrix} G & \overline{G} \\ \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_1^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_g^T \\ c_1 & \cdots & c_g & \mathbf{P}_{22} \end{bmatrix}
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Perron Complements

\(p_{11} \cdots p_{gg}\) are \(1 \times 1\) \(\implies\) Perron complements = 1

\(\implies\) Perron vectors = 1

One significant complement \(\mathbf{S}_2 = \mathbf{P}_{22} + \mathbf{P}_{21}(\mathbf{I} - \mathbf{P}_{11})^{-1} \mathbf{P}_{12}\)

One significant Perron vector \(\mathbf{s}_2^T \mathbf{S}_2 = \mathbf{s}_2^T\)
Aggregation

Partition

\[
P_{n \times n} = \frac{G}{G} \begin{pmatrix} G & \overline{G} \\ P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{bmatrix} p_{11} & \cdots & p_{1g} & r_{1}^T \\ \vdots & \ddots & \vdots & \vdots \\ p_{g1} & \cdots & p_{gg} & r_{g}^T \\ c_{1} & \cdots & c_{g} & P_{22} \end{bmatrix}
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\pi^T = (\pi_1, \ldots, \pi_g \mid \pi_{g+1}, \ldots, \pi_n)
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Perron Complements

$p_{11} \cdots p_{gg}$ are $1 \times 1$ \implies Perron complements = 1

\implies Perron vectors = 1

One significant complement \( S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12} \)

One significant Perron vector \( s_2^T S_2 = s_2^T \)

A/D corollary \( \implies s_2^T = (\pi_{g+1}, \ldots, \pi_n) / \sum_{i=g+1}^{n} \pi_i \)
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\((\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)\) (the smaller ones)

By A/D Corollary

\[ S_2^T = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^n \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^n \phi_i} \equiv \tilde{S}_2^T \]
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\[
(\pi_{g+1}, \ldots, \pi_n) \approx (\phi_{g+1}, \ldots, \phi_n)
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(the smaller ones)

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\mathbf{s}_2^T = \frac{(\pi_{g+1}, \ldots, \pi_n)}{\sum_{i=g+1}^{n} \pi_i} \approx \frac{(\phi_{g+1}, \ldots, \phi_n)}{\sum_{i=g+1}^{n} \phi_i} \equiv \tilde{\mathbf{s}}_2^T
\]

Approximate Aggregation Matrix

\[
\tilde{\mathbf{A}} \equiv \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \mathbf{e} \\ \tilde{s}_2^T \mathbf{P}_{21} & \tilde{s}_2^T \mathbf{P}_{22} \mathbf{e} \end{bmatrix}_{g+1 \times g+1}
\]
Approximate Aggregation

Some Old PageRanks Approximate New Ones

\[
\left( \pi_{g+1}, \ldots, \pi_n \right) \approx \left( \phi_{g+1}, \ldots, \phi_n \right)
\]

(the smaller ones)

By A/D Corollary

\[
\tilde{s}_2^T = \frac{\left( \pi_{g+1}, \ldots, \pi_n \right)}{\sum_{i=g+1}^{n} \pi_i} \approx \frac{\left( \phi_{g+1}, \ldots, \phi_n \right)}{\sum_{i=g+1}^{n} \phi_i} \equiv \tilde{s}_2^T
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Approximate Aggregation Matrix

\[
\tilde{A} \equiv \begin{bmatrix}
P_{11} & P_{12}e \\
\tilde{s}_2^T P_{21} & \tilde{s}_2^T P_{22}e
\end{bmatrix}_{g+1 \times g+1}
\]

\[
\tilde{\alpha}^T = \left( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_g, \tilde{\alpha}_{g+1} \right)
\]

By A/D Theorem

\[
\tilde{\pi}^T \equiv \left( \tilde{\alpha}_1, \ldots, \tilde{\alpha}_g \mid \tilde{\alpha}_{g+1} \tilde{s}_2^T \right) \approx \pi^T
\]

(not bad)
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges

Solution

Perturb A/D output to move off of fixed point
Move in direction of solution

\[ \tilde{\pi}^T = \tilde{\pi}^T P \]

(a smoothing step)
Iterative Aggregation

Improve By Successive Aggregation / Disaggregation?

NO! Can’t do twice — fixed point emerges

Solution

Perturb A/D output to move off of fixed point
Move in direction of solution
\[ \tilde{\pi}^T = \tilde{\pi}^T P \]

(a smoothing step)

The Iterative A/D Updating Algorithm

Determine the “\(G\)-set” partition \( S = G \cup \overline{G} \)
Approximate A/D step generates \( \tilde{\pi}^T \)
Smooth \( \tilde{\pi}^T = \tilde{\pi}^T P \)
Use \( \tilde{\pi}^T \) as input to another approximate aggregation step

...
Convergence

**THEOREM**

Always converges to the new PageRank vector $\pi^T$
Convergence

THEOREM

Always converges to the new PageRank vector $\pi^T$

Converges for all partitions $S = G \cup \overline{G}$
Convergence

**THEOREM**

Always converges to the new PageRank vector $\pi^T$

Converges for all partitions $S = G \cup \overline{G}$

Rate of convergence governed by $|\lambda_2(S_2)|$

$$S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$$
Convergence

THEOREM

Always converges to the new PageRank vector $\pi^T$

Converges for all partitions $S = G \cup \overline{G}$

Rate of convergence governed by $|\lambda_2(S_2)|$

$S_2 = P_{22} + P_{21}(I - P_{11})^{-1}P_{12}$

THE GAME

Find a relatively small $G$ to minimize $|\lambda_2(S_2)|$
Experiments

Test Networks From Crawl Of Web (Supplied by Ronny Lempel)

Censorship
562 nodes 736 links

Movies
451 nodes 713 links

MathWorks (Supplied by Cleve Moler)
517 nodes 13,531 links

Abortion
1,693 nodes 4,325 links

Genetics
2,952 nodes 6,485 links

California
9,664 nodes 16,150 links
The Updates

- # Nodes Added = 3
- # Nodes Removed = 50
- # Links Added = 10
- # Links Removed = 20

(Different values have little effect on results)

Stopping Criterion

1-norm of residual < 10^{-10}
### Movies

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterations</th>
<th>Time</th>
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<tbody>
<tr>
<td></td>
<td>17</td>
<td>.40</td>
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</tbody>
</table>

| Iterative Aggregation | $|G|$ | Iterations | Time |
|------------------------|-----|------------|------|
|                        | 5   | 12         | .39  |
|                        | 10  | 12         | .37  |
|                        | 15  | 11         | .36  |
|                        | 20  | 11         | .35  |
|                        | 100 | 9          | .33  |
|                        | 200 | 8          | .35  |
|                        | 300 | 7          | .39  |
|                        | 400 | 6          | .47  |

\[\text{nodes} = 451 \quad \text{links} = 713\]
## Movies

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*nodes = 451   links = 713*
## Power Method

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<td>38</td>
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</table>

## Iterative Aggregation

| $|G|$ | Iterations | Time |
|----|------------|------|
| 5  | 38         | 1.68 |
| 10 | 38         | 1.66 |
| 15 | 38         | 1.56 |
| 20 | 20         | 1.06 |
| 25 | 20         | 1.05 |
| 50 | 10         | .69  |
| 100| 8          | .55  |

| 300| 6 | .65 |
| 400| 5 | .70 |

*nodes = 562*,  *links = 736*
Censorship

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| Iterative Aggregation | \(|G|\) | Iterations | Time |
|-----------------------|--------|------------|------|
|                       | 5      | 38         | 1.68 |
|                       | 10     | 38         | 1.66 |
|                       | 15     | 38         | 1.56 |
|                       | 20     | 20         | 1.06 |
|                       | 25     | 20         | 1.05 |
|                       | 50     | 10         | 0.69 |
|                       | 100    | 8          | 0.55 |
|                       | 200    | 6          | 0.53 |
|                       | 300    | 6          | 0.65 |
|                       | 400    | 5          | 0.70 |

\(\text{nodes} = 562 \quad \text{links} = 736\)
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<tr>
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<td>52</td>
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<td></td>
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<tr>
<td>15</td>
<td>52</td>
<td>1.23</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>42</td>
<td>1.05</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>20</td>
<td>1.13</td>
<td></td>
</tr>
<tr>
<td>300</td>
<td>11</td>
<td>.83</td>
<td></td>
</tr>
<tr>
<td>400</td>
<td>10</td>
<td>1.01</td>
<td></td>
</tr>
</tbody>
</table>

 nodes = 517   links = 13,531
## MathWorks

### Power Method

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>54</td>
<td>1.25</td>
</tr>
</tbody>
</table>

### Iterative Aggregation

| $|G|$ | Iterations | Time  |
|-----|------------|-------|
| 5   | 53         | 1.18  |
| 10  | 52         | 1.29  |
| 15  | 52         | 1.23  |
| 20  | 42         | 1.05  |
| 25  | 20         | 1.13  |
| 50  | 18         | .70   |
| 100 | 16         | .70   |
| 200 | 13         | .70   |
| 300 | 11         | .83   |
| 400 | 10         | 1.01  |

*nodes = 517, links = 13,531*
### Power Method

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>106</td>
<td>37.08</td>
</tr>
</tbody>
</table>

### Iterative Aggregation

| $|G|$ | Iterations | Time  |
|-----|------------|-------|
| 5   | 109        | 38.56 |
| 10  | 105        | 36.02 |
| 15  | 107        | 38.05 |
| 20  | 107        | 38.45 |
| 25  | 97         | 34.81 |
| 50  | 53         | 18.80 |
| 250 | 12         | 5.62  |
| 500 | 6          | 5.21  |
| 750 | 5          | 10.22 |
| 1000| 5          | 14.61 |

Nodes = 1,693   Links = 4,325
<table>
<thead>
<tr>
<th>Abortion</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Power Method</strong></td>
</tr>
<tr>
<td>Iterations</td>
</tr>
<tr>
<td>106</td>
</tr>
<tr>
<td>109</td>
</tr>
<tr>
<td>15</td>
</tr>
<tr>
<td>20</td>
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<td>25</td>
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<tr>
<td>50</td>
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<tr>
<td>100</td>
</tr>
<tr>
<td>250</td>
</tr>
<tr>
<td>500</td>
</tr>
<tr>
<td>750</td>
</tr>
<tr>
<td>1000</td>
</tr>
</tbody>
</table>

\(\text{nodes} = 1,693\quad \text{links} = 4,325\)
# Genetics

## Power Method

<table>
<thead>
<tr>
<th>Iterations</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td>92</td>
<td>91.78</td>
</tr>
</tbody>
</table>

## Iterative Aggregation

| \(| G | \) | Iterations | Time  |
|------|------------|-------|
| 5    | 91         | 88.22 |
| 10   | 92         | 92.12 |
| 20   | 71         | 72.53 |
| 50   | 25         | 25.42 |
| 100  | 19         | 20.72 |
| 250  | 13         | 14.97 |
| 1000 | 5          | 17.76 |
| 1500 | 5          | 31.84 |

\(\text{nodes} = 2,952 \quad \text{links} = 6,485\)
### Genetics

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterative Aggregation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Iterations</strong></td>
<td><strong>Time</strong></td>
</tr>
<tr>
<td>92</td>
<td>91.78</td>
</tr>
<tr>
<td>10</td>
<td>92</td>
</tr>
<tr>
<td>20</td>
<td>71</td>
</tr>
<tr>
<td>50</td>
<td>25</td>
</tr>
<tr>
<td>100</td>
<td>19</td>
</tr>
<tr>
<td>250</td>
<td>13</td>
</tr>
<tr>
<td>500</td>
<td>7</td>
</tr>
<tr>
<td>1000</td>
<td>5</td>
</tr>
<tr>
<td>1500</td>
<td>5</td>
</tr>
</tbody>
</table>

\[\text{nodes} = 2,952 \quad \text{links} = 6,485\]
California

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterative Aggregation</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Iterations</strong></td>
<td><strong>Time</strong></td>
</tr>
<tr>
<td><strong>176</strong></td>
<td><strong>5.85</strong></td>
</tr>
<tr>
<td><strong>500</strong></td>
<td><strong>19</strong></td>
</tr>
<tr>
<td><strong>1000</strong></td>
<td><strong>15</strong></td>
</tr>
<tr>
<td><strong>1250</strong></td>
<td><strong>20</strong></td>
</tr>
<tr>
<td><strong>2000</strong></td>
<td><strong>13</strong></td>
</tr>
<tr>
<td><strong>5000</strong></td>
<td><strong>6</strong></td>
</tr>
</tbody>
</table>

**nodes** = 9,664  **links** = 16,150
## California

<table>
<thead>
<tr>
<th>Power Method</th>
<th>Iterations</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>176</td>
<td>5.85</td>
</tr>
</tbody>
</table>

| Iterative Aggregation | $|G|$ | Iterations | Time |
|-----------------------|------|------------|------|
|                       | 500  | 19         | 1.12 |
|                       | 1000 | 15         | 0.92 |
|                       | 1250 | 20         | 1.04 |
|                       | 1500 | 14         | 0.90 |
|                       | 2000 | 13         | 1.17 |
|                       | 5000 | 6          | 1.25 |

$nodes = 9,664 \quad links = 16,150$
“L” Curves

censorship

censorship

movies

movies

mathworks

mathworks

abortion

abortion

genetic

genetic

CA

CA

movies

movies

mathworks

mathworks

abortion

abortion

genetic

genetic

CA

CA
“L” Curves

censorship

movies

mathworks

abortion

genetic

CA
Comparisons

Race

→ Power Method

→ Power Method + Quadratic Extrapolation

→ Iterative Aggregation

→ Iterative Aggregation + Quadratic Extrapolation
Comparisons

Race
→ Power Method
→ Power Method + Quadratic Extrapolation
→ Iterative Aggregation
→ Iterative Aggregation + Quadratic Extrapolation

NC State Internal Crawl
→ 10,000 nodes + 101,118 links
  → 50 nodes added
  → 30 nodes removed
    → 300 links added
    → 200 links removed
Iterations

![Graph showing the relationship between iteration and log10 residual. The graph plots power method against iteration.]
Iterations

- Power method
- Power method with quad. extrap.
- IAD
- IAD with quad. extrap.
## Timings

| Power                   | Iterations | Time (sec) | \(|G|\) |
|-------------------------|------------|------------|--------|
| Power+Quad              | 162        | 9.69       |        |
| IAD                     |            |            |        |
| IAD+Quad                |            |            |        |

\(\text{nodes} = 10,000\) \quad \text{links} = 101,118
|          | Iterations | Time (sec) | $|G|$ |
|----------|------------|------------|------|
| Power    | 162        | 9.69       |      |
| Power+Quad | 81        | 5.93       |      |
| IAD      |            |            |      |
| IAD+Quad |            |            |      |

$nodes = 10,000 \quad links = 101,118$
## Timings

|       | Iterations | Time (sec) | $|G|\$ |
|-------|------------|------------|------|
| Power | 162        | 9.69       |      |
| Power+Quad | 81        | 5.93       |      |
| IAD   | 21         | 2.22       | 2000 |
| IAD+Quad |          |            |      |

$nodes = 10,000 \quad links = 101,118$
## Timings

| Model       | Iterations | Time (sec) | $|G|$ |
|-------------|------------|------------|-----|
| Power       | 162        | 9.69       |     |
| Power+Quad  | 81         | 5.93       |     |
| IAD         | 21         | 2.22       | 2000|
| IAD+Quad    | 16         | 1.85       | 2000|

$nodes = 10,000 \quad links = 101,118$
Conclusion

- Iterative A/D with appropriate partitioning and smoothing shows promise for updating Markov chains with power law distributions